MATH-505A: Homework # 2

Due on Friday, September 5, 2014

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Exercise # 1.5

(1)

Given: A, B are independent
To Prove: (A^C, B); (A^C, B^C) are independent

Since A, B are independent:

$$P(A \cap B) = P(A)(B) \tag{1}$$

Thus,

$$P(A \cap B) = (1 - P(A^C))P(B) = P(B) - P(B)P(A^C) \tag{2}$$

Rearranging 2:

$$P(B)P(A^C) = P(B) - P(A \cap B) \tag{3}$$

$P(B) - P(A \cap B)$ signifies **'in B but not in $A\ AND\ B'$.** Thus, it should belong to $A^C\ AND\ B$

$$P(B) - P(A \cap B) = P(A^C \cap B) = P(A^C)P(B) \tag{4}$$

From 3 and 4 : $A^C.B$ are independent.
Similiary to prove A^C, B^C are independent, we perform substitute B^C in $P(B \cap A^C)$, since 4 is true to obtain:

$$P(B^C \cap A^C) = P(A^C)P(B^C) \tag{5}$$

(2)

$A_{ij} = i^{th}$ and j^{th} rolls produce the same number.
For any $i \neq j$, total outcome are $6 * 6 = 36$ and number of favourable outcomes are $\binom{6}{1} * 1 = 6$, thus $p(A_{ij}) = \frac{6}{36} = \frac{1}{6}$
Consider $P(A_{ij} \cap A_{kj})$, such that $i \neq j \neq k$, then :

$P(A_{ij} \cap A_{jk})$ refers to the probability when i^{th}, j^{th} and k^{th} rolls show the same number, which can be caluclated as: $P(A_{ij} \cap A_{kj}) = \frac{\binom{6}{1}*1*1}{6*6*6} = \frac{1}{36} = P(A_{ij})P(A_{jk})$
Thus, A_{ij} are pairwise independent as it is true for any choice of i, j, k as long as $i \neq j \neq k$
Consider:

$$P(A_{ij} \cap A_{jk} \cap A_{kl}) = \frac{\binom{6}{1}*1*1*1}{6*6*6} = \frac{1}{36} \neq P(A_{ij})P(A_{jk})P(A_{kl})$$

Since $P(A_ij \cap A_jk \cap A_kl) \neq P(A_{ij})P(A_{jk})P(A_{kl})$, it will not be true in general.
And since the independence criterion is not satisfied for the above case, it will not be true for the case when A_{ij} are considered togethter for all values of i, j.

(3)

To Prove:
(a) outcomes of coin tosses are independent
(b) Given a sequence of length m of heads and tails the chance of it occuring in first m tosses i s 2^-m.

In order to prove (a) and (b) are equivalent it is sufficient to prove that if $a \implies b$ and $b \implies a$.
If the outcomes are independent, probability of a head or tail in a sequence is $\frac{1}{2}$. Consider m tosses, since they are independent the probability of seeing any string of H and T is given by $\frac{1}{2} * \frac{1}{2} * ... * (m) times = \frac{1}{2^m} = 2^{-m}$. Hence $\implies b$.
Now consider if b is true, then : P(m) = $2^{-m} \implies P(m+1) = 2^{-(m+1)}$, . The $P(m+1)$ case is similar to $P(m)$ with an extra toss. $P(m+1) = 2^{-m} * \frac{1}{2}$. The extra half factor is accounted by the extra toss that is performed which must be independent with respect to the m tosses for yielding such a relation for $P(m+1) \implies a$ is true and this is true for any m. This inductive hypothesis proves that the tosses must be indepedent($P(1) = \frac{1}{2}$ is trivially true)
and hence $a \impliedby b$ and $a \implies b$ so a and b are equivalent statements.

(4)

Given: $\Omega = \{1, 2, 3, ...p\}$ where p is prime. F is set of all subsets of ω; $P(A) = \frac{|A|}{p}$ **To Prove:** A 'or' B is a null set or is the set $Omega$
$P(A) = \frac{|A|}{p}$
$P(B) = \frac{|B|}{p}$
Now, from the definition of $P(A)$:

$$P(A \cap B) = \frac{|A \cap B|}{p} \tag{6}$$

Since A, B are independent:

$$P(A \cap B) = P(A)P(B) = \frac{|A \cap B|}{p} = \frac{|A|}{p}\frac{|B|}{p} \tag{7}$$

Thus:

$$p * |A \cap B| = |A||B| \implies |A||B| \ mod \ p = 0 \tag{8}$$

where mod operator gives the remainder.
and

$$0 \le |A|, |B| \le p \ AND \ |A||B| \ mod \ p = 0 \implies (|A| \ or |B| = p) \ OR \ (|A| \ OR \ |B| = 0) \implies \text{A,B are either null or complete sets(with } |A|, |B| = |\Omega|\text{).}$$

(5)

Given:

$$P(A,B|C) = P(A|C)P(B|C) \tag{9}$$

for all A,B.
If A, B are independent :

$$P(A|B) = P(A) \tag{10}$$

$$P(A,B|C) = \frac{P(A,B,C)}{P(C)} = P(A|B,C) * \frac{P(B,C)}{P(C)} = P(A|B,C) * P(B|C) \tag{11}$$

Hence the conditional independence of A and B given C is dependent on conditional independence of A given B and C and B given C and neither implies $P(A|B) = P(A)$ nor is implied by this.

Part2:

For which event C, are A and B(for all A,B) independent **iff** they are conditionaly independent given C: ? If $P(A,B|C) = P(A|C) * P(B|C) \implies P(A|B) = P(A)$ then what is C?. This relation can be set to true for all values of A,B if $P(C)$ is set to 1. because that would automatically imply $P(A,B) = P(A)P(B)$

Thus $P(C) = 1$

Exercise # 1.5 continued on next page...

(7)

$A = \{$all children of same sex $\}$
$B = \{$ there is at most one boy $\}$
$C = \{$ one boy and one girl included $\}$

$P(A) = P($ all boys $) + P($all girls$) = (\frac{1}{2} * \frac{1}{2} * \frac{1}{2}) * 2 = \frac{1}{4}$

$P(B) = P($ 0 boys $) + P(1boy) = (\frac{1}{8}) + 3 * (\frac{1}{8}) = \frac{1}{2}$

$P(C) = P($1 boy + 1 girl $) = 2 * (\frac{1}{2} * \frac{1}{2} * \frac{1}{2}) * 3 = \frac{3}{4}$

Part a): A is independent of B and B is independent of C
$P(A \cap B) =$ all chidren are of same sex AND there is at most one boy $\implies$ all children are boys
$\implies P(A \cap B) = \frac{1}{2} * \frac{1}{2} * \frac{1}{2} = \frac{1}{8} = P(A) * P(B)$.
Hence A,B are independent

$P(B \cap C) =$ there is at most one boy AND there is one boy and a girl $\implies$ there is one boy and two girls.
$P(B \cap C) = \frac{1}{2} * \frac{1}{2} * \frac{1}{2} * 3 = \frac{3}{8} = P(B) * P(C)$
Hence B,C are independent

Part b): Is A independednt of C?
$P(A \cap C) =$ the family includes boy and girl AND all children are of same sex

Clearky $P(A \cap B) = \phi$ and hence A,B are NOT necessarily independent!(null intersection does not imply independence)
Part c): Do the results hold if boys and girls are not equally likely?
NO. if the girls and boys are not likely these events will have different probabilitie
Part d): Do these results hold if there are 4 children?
No, the results might not necessarily be valid for 4 children as the numberof permutations would be tbe