MATH-505A: Homework # 5

Due on Friday, September 26, 2014

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Contents

Exercise # 2.7

(1)

Coin toss shows head with probability $= p$ **To Find:** $P(X > m)$
By definition $P(X > m) = 1 - P(X < m)$
$P(X < m) = P(head\ comes\ on\ toss\ 1) + P(head\ comes\ on\ toss\ 2) + .. + P(head\ comes\ on\ toss\ m-1) =$
$p + (1-p)p + (1-p)^2 p + ... + (1-p)^(m-1)p = p(\frac{1-(1-p)^{m-1+1}}{1-(1-p)}) = 1 - (1-p)^m$
Thus $P(X > m) = 1 - 1 - (1-p)^m = (1-p)^m$
The distribution function of X is given by:
$$F(x) = \begin{cases} 1-(1-p)^x & x \leq 0 \\ 0 & otherwise \end{cases}$$
given that x takes only integer values[X is a discrete random variable]

(2 a)

X is a random variable. Let $\Omega = \{x1, x2, ..., x_n\}$ Define a indicator random variables $I_j(x_j)$ such that:
$$I_j(x) = \begin{cases} 1 & if x = x_j \\ 0 & otherwise \end{cases}$$
Thus X can be now expressed as :
$X = \sum_{j=1}^{n} x_j I_j(x_j)$

(2 c)

Consider the set of events $X = \{X_1, X_2, X_3, ..., X_n\}$ such that $X_1(\omega) < X_2(\omega) < ... < X_m(\omega) \forall \omega in \Omega$
In order to prove if X is a random variable we consider $\{X(\omega) \leq x\}$ which is equivalent to $\{X_i(\omega) \leq x\} \forall i \in [1, n]]$ which is equivalent to $\{min X(\omega) \leq x\}$ where $min X$ refers to $min(X_i)$

Exercise # 2.7 continued on next page...

(4)

$F(x) = \begin{cases} 0 & if x < 0 \\ \frac{1}{2}x & if 0 < x < 2, \\ 1 & if x > 2 \end{cases}$

Thus $f(x) = F'(x)$:

$f(x) = \begin{cases} 0 & if x < 0 \\ \frac{1}{2} & if 0 < x < 2, \\ 0 & if x > 2 \end{cases}$

Part a: $P(\frac{1}{2} \leq X \leq \frac{3}{2})$ $P(\frac{1}{2} \leq X \leq \frac{3}{2}) = \int_{\frac{1}{2}}^{\frac{3}{2}} f(x)dx = \int_{\frac{1}{2}}^{\frac{3}{2}} \frac{1}{2}dx = \frac{1}{2}$

Part b: $P(1 \leq X \leq 2)$ $P(1 \leq X \leq 2) = \int_1 2\frac{1}{2}dx = \frac{1}{2}$

Part c: $P(Y \leq X)$ $P(Y \leq X) = P(X^2 \leq X) = P(X^2 - X \leq 0) = P(X^2 - X \leq 0) = P(X(X-1) \leq 0)$
Since $X(X-1) \leq 0 \implies 0 \leq X \leq 1$

Thus, $P(Y \leq X) = P(0 \leq X \leq 1) = \int_0^1 \frac{1}{2}dx = \frac{1}{2}$

Part d: $P(Y \leq 2X)$ $P(Y \leq 2X) = p(X \leq 2X^2) = P(X(1-2X) \leq 0) = P(X(2X-1) \geq 0) = P(X \geq \frac{1}{2}) \cup P(X \leq 0) = P(X \geq \frac{1}{2}) + 0 = \int_{\frac{1}{2}}^2 \frac{1}{2}dx + \int_2^\infty 0dx = \frac{3}{4}$

Part e: $P(X + X^2 \leq \frac{3}{4})$ $P(X + X^2 \leq \frac{3}{4}) = P((X+0.5)^2 \leq 1) = P(X \leq 0.5) = \int_{0.5}^1 \frac{1}{2} = \frac{1}{4}$

Part f: $P(\sqrt{X} \leq z)$ $P(\sqrt{X} \leq z) = P(X \leq z^2) = \frac{1}{2}z^2\ if 0 \leq z \leq \sqrt{(2)}$ 0 otherwise

Exercise # 2.7 continued on next page...

(5)

$$F(x) = \begin{cases} 0 & if x < -1 \\ 1-p & if -1 \leq x < 0, \\ 1-p+\frac{1}{2}xp & if 0 \leq x < 2 \\ 1 & x \geq 2 \end{cases}$$

Thus $f(x) = F'(x)$:

$$f(x) = \begin{cases} 0 & if x < -1 \\ 0 & if -1 \leq x < 0, \\ \frac{1}{2}p & if 0 \leq x < 2 \\ 0 & x \geq 2 \end{cases}$$

Part a $P(X=-1)$ $P(X=-1) = 1-p$
Part b $P(X=0)$
$P(X=0) = f(x=0) = 0$

Part c$P(X \geq 1)$
$P(X \geq 1) = \int_1^2 (\frac{1}{2}p)dx + \int_1 0dx = \frac{1}{2}p$

(7)

p(Teeny Weeny is overbooked) = p(All 10 passengers turn up) = $(\frac{9}{10})^10 = 0.34$
p(Blockbuster airways is overbooked) = p(19 or 20 passengers turn up) = $\binom{20}{19}(\frac{1}{10})(\frac{9}{10})^19 + \binom{20}{2}(\frac{1}{10})^2(\frac{9}{10})^18 = .39$
Thus blockbuster ariways is overbooked on average.

(9)

Part a
$X^+ = max(0, X)$:
As $F(x)$ is positive definite:
$P(X^+ \leq x) = \begin{cases} 0 & x \leq 0 \\ F(x) & x \geq 0 \end{cases}$ **Part b**
For $X^- = -min(0, X)$:
$P(X^- \leq x) = P((0 \leq x) \cup (-X \leq 0)) = P((0 \leq x) \cup (X \geq -x)) = P(0 \leq x) + P(X \geq -x) = P(0 \leq x) + (1 - P(X \leq -x))$
Thus, $P(X^- \leq x) = \begin{cases} 0 & x \leq 0 \\ 1 - F(-x) & x \geq 0 \end{cases}$
Part c

$|X| = X^+ + X^-$
Thus, $P(|X| \leq x) = P(-x \leq X \leq x) = F(x) - F(-x)$
Part d
$-X$
$P(-X \leq x) = P(-X \geq x) = 1 - P(X \leq -x) = 1 - F(-x)$

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Given: Median $m = \lim_{y \to \infty} F(y) \leq \frac{1}{2} \leq F(m)$ **To Prove:** Every distribution of F has a median and it is a closed interval
Since F is a continuous function with range $[0, 1]$ applying intermediate values theorem:
F will take all values between its extremum. so every distribution of F has a median and is closed under $[0, 1]$

(12)

Consider the outcomes of the dice to be $X1, X2$ and the sum to be $S = X1 + X2$. Let us assume the outcomes that $S = 2$ to $S = 12$ are equally likely Then let:
$S = 2 : A = \{(1, 1)\}$
$S = 9 : A = \{(3, 6), (4, 5), (5, 4), (6, 3)\}$
Thus for $S = 2, 9$, $A = 1, 4$ the number of favourable events are unequal $\implies$ the initial assumption is false!

(15)

(18)

Total ways to arrange 8 pawns on a chessboard = $\binom{64}{8}$
Number of ways of being in straight line = Number of ways they are in same row + Number of ways they are in same column + Number of ways they are in same diagonal = $\binom{8}{1} + \binom{8}{1} + \binom{2}{1}$ p(pawns are in straight line) = $\frac{\binom{8}{1}+\binom{8}{1}+\binom{2}{1}}{\binom{64}{8}} = \frac{18}{\binom{64}{8}}$
Number of ways that pawns are not in the same row or column = First pawn has 8 rows/columns to choose from, then the second pawn will have 7 rows/columns to choose from and so on. At each step the row taken by the pawn at earlier step is removed from choice of consideration
Thus Number of ways that pawns are not in the same row or column = $8 * 7 * 6... = 8!$
Thus p(Number of ways that pawns are not in the same row or column) = $\frac{8!}{\binom{64}{8}}$

(19)

Part a

$F(x) = \begin{cases} 1 - e^{-x^2} & x \geq 0 \\ 0 & otherwise \end{cases}$ $F(\infty) = 1$ and $F(-\infty) = 0$

Differentiating $F(x)$ we get:

$f(x) = \begin{cases} 2xe^{-x^2} & x \geq 0 \\ 0 & otherwise \end{cases}$ F is a distribution function

Part b

$F(x) = \begin{cases} e^{-\frac{1}{x}} & x > 0 \\ 0 & otherwise \end{cases}$ $F(\infty) = 1$ and $F(-\infty) = 0$

Differentiating $F(x)$ we get:

$f(x) = \begin{cases} \frac{e^{-\frac{1}{x}}}{x^2} & x > 0 \\ 0 & otherwise \end{cases}$ F is a distribution function

Part c

$F(x) = \begin{cases} \frac{e^x}{e^x+e^{-x}} & x \geq 0 \\ 0 & otherwise \end{cases}$ $F(\infty) = 1$ and $F(-\infty) = 0$

Differentiating $F(x)$ we get:

$f(x) = \begin{cases} \frac{2}{(e^x+e^-x)^2} & x \geq 0 \\ 0 & otherwise \end{cases}$ F is a distribution function

Part d

$F(x) = \begin{cases} e^{-x^2} + \frac{e^x}{e^x+e^{-x}} & x \geq 0 \\ 0 & otherwise \end{cases}$ $F(\infty) = 1$ and $F(-\infty) = 0$

Differentiating $F(x)$ we get:

$f(x) = \begin{cases} -2xe^{-x^2} + \frac{2}{(e^x+e^-x)^2} & x \geq 0 \\ 0 & otherwise \end{cases}$ F is NOT a distribution function as f can take in negative values.