MATH-505A: Homework # 6

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Exercise # 3.1

1

Part a: $f(x) = C2^{-x}$
For f(x) to be a mass function $\sum_1^\infty C2^-i = 1$ $C(\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + ...) = C\frac{1}{2} * \frac{1}{1-\frac{1}{2}} = 1 \implies C = 1$
Part b: $f(x) = \frac{C2^{-x}}{x}$ $\sum_1^\infty \frac{C}{2^i i} = 1$
Notice $ln(1+x) = x + \frac{x^2}{2} + \frac{x^3}{3} + ...$
Hence $C\sum_1^\infty \frac{(\frac{1}{2})^i}{i} = C\ ln(1+1/2) = 1 \implies C = \frac{1}{ln1.5}$
Part c: $f(x) = Cx^-2$ $\sum_1^\infty \frac{C}{x^2} = 1$
Besel sum:
$\sum_1^n \frac{1}{i^2} =$
Using taylor expansion of $sinx$ and the fact that $\frac{sinx}{x}$ has roots at $x = \pi, 2\pi, 3\pi.....$:

$\frac{sinx}{x} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} + ... = (1-\frac{x}{\pi})(1-\frac{x}{2\pi})(1-\frac{x}{3\pi})....(1-\frac{x}{\pi})$
The product of productions of $\frac{sinx}{x}$ is given the the coefficient of x^2 in the original and hence $-\frac{1}{3!} = -\pi^2\sum_1^\infty i^2$
Thus $\sum_1^\infty \frac{C}{x^2} = C * \frac{\pi^2}{6}$. Thus C=$\frac{6}{\pi^2}$
Part d: $C2^x/x!$ $\sum_1^\infty C2^i/i! = C\sum_1^\infty 2^i/i! = Ce^2 \implies C = \frac{1}{e^2}$

2 (i)

Part a $P(X > 1) = \sum_2^\infty 2^{-i} = \frac{1}{4} * 2 = \frac{1}{2}$
Part b $P(X > 1) = \frac{1}{1.5}\sum_2^\infty \frac{(\frac{1}{2})^i}{i} = 1 - ln(1.5)/2$
Part c $P(X > 1) = 1 - \frac{6}{\pi^2}1^{-2} = 1 - \frac{6}{\pi^2}$
Part d $P(X > 1) = 1 - \frac{1}{e^2}2 = 1 - \frac{2}{e^2}$

2 (ii)

All functions in a,b,c,d are decreasing. Hence the most probable value occurs at $x = 1$

2 (iii)

Probability that X is even = $P(X = 2k)$ for $k = 1, 2, 3...$
Part a $P(X = 2k) = 2^{-2k}$ Summing up over all k: $P = \frac{1}{2^2} + \frac{1}{2^4} + \frac{1}{2^6} + ... = \frac{1}{4}\frac{1}{1-\frac{1}{4}} = \frac{1}{3}$
Part b $P(X = 2k) = ln(3.5)\frac{\frac{1}{2}^{2k}}{2k} = ln(3.5)\frac{1}{2}\frac{(\frac{1}{4})^k}{k} = \frac{ln(3.5)}{2}e^{1.25}$
Part c
$P(X = 2k) = \frac{6}{\pi^2}4k^2 = \frac{3}{2\pi^2} * \frac{\pi^2}{6} = \frac{1}{4}$
Part d
$P(X = 2k) = \frac{1}{e^2}\frac{2^{2k}}{(2k)!}$

3

Since the coin tosses are independent, the choice can be represented by two successive coin tosses with probability of heads being $p * p$. Thus $P(X = k) = \binom{n}{k}p^{2k}(1-p^2)^{n-k}$

5a

For Binomial: $f(k) = \binom{n}{k}p^k(1-p)^{n-k}$ Consider $LHS = f(k-1)*f(k+1)$ $LHS = \binom{n}{k-1}p^{k-1}(1-p)^{n-k-1} + \binom{n}{k+1}p^{k+1}(1-p)^{n-k-1} = \binom{n}{k-1}\binom{n}{k+1}(p^k(1-p)^{n-k})^2$ $RHS = \binom{n}{k}^2$ We now focus on $\binom{n}{k-1}\binom{n}{k+1}$ Let $y = \frac{\binom{n}{k-1}\binom{n}{k+1}}{\binom{n}{k}^2}$ Expanding: $y = \frac{n!n!(n-k)!(n-k)!k!k!}{(k-1)!(k+1)!(n-k+1)!(n-k-1)!} = \frac{k(n-k)}{(k+1)(n-k+1)}$ $\frac{k}{k+1} \leq 1$ and $\frac{n-k}{n-k+1} \leq 1$ $\forall k$

Hence $y \leq 1$

Thus $LHS \leq RHS$

For Poisson $f(k) = e^{-\lambda}\frac{\lambda^k}{k!}$ $LHS = f(k-1)f(k+1) = \frac{e^{-2\lambda}\lambda^{2k}}{(k+1)!(k-1)!}$

Thus $LHS \leq RHS$

5b

$f(k) = \frac{90}{(\pi k)^4}$ $LHS = f(k+1)f(k-1) = \frac{90^2}{\pi^8(k+1)^4(k-1)^4}$ $RHS = f(k+1)^2 = \frac{90^2}{\pi^8(k)^8}$

$y = LHS/RHS = \frac{k^8}{(k+1)^4(k-1)^4} = (\frac{k}{(k+1)}\frac{k}{(k-1)})^4 = (\frac{k^2}{k^2-1})^4 \geq 1$

Thus $LHS \geq RHS$

5c

Any function of the form $P(x=k) = \frac{1}{n}$ satisfies $f(k)f(k-1) = f(k)^2$ Note: We aren't explicitly talking about countably many case.

Exercise # 3.2

2

Part a $P(min(X,Y) \leq x) = 1 - P(X > x \cup Y > y) = 1 - P(X > x)P(Y > y) = 1 - 4^{-x}$

Part b $P(Y > x) = \frac{1}{3}$ by symmetery of $P(X > Y) = P(X < Y) = P(X = Y)$ **Part c** $P(X = Y) = \frac{1}{3}$ as in (b)

Part d $P(X \geq y) = \frac{1}{3}$ as in (b)

Part e $P(X\ dividesY) = P(Y = kX) = P(Y = kx, X = x) = P(Y = kx)P(X = x) = \sum 2^{-kx}2^{-x} = \sum_{k=1} k = \infty \frac{1}{2^{k+1}-1} =$

Part f $P(X = rY) = P(X = ry, Y = y) = \sum 2^{-ry}2^{-y} = 2^{-r-1}(2) = 2^{-r}$

4

Consider three possibilities: 1. A rolls a 6, B,C do not

2. A and B roll a 6

3. No one rolls 6

$p = \frac{1}{6}(\frac{5}{6})^2 P(B < C) + \frac{1}{6}\frac{1}{6} + (\frac{5}{6})^3 p$

$P(B < C) = \frac{5}{6}\frac{5}{6}P(B < C) + \frac{1}{6} \implies P(B < C) = \frac{6}{11}$

$p(1 - \frac{125}{216}) = \frac{25}{216}\frac{6}{11} + \frac{6}{216}$

$p(\frac{91}{216}) = \frac{216}{216*11} = \frac{216}{1001}$

Exercise # 3.3

1

$E(\frac{1}{X}) = \sum p(x)(\frac{1}{x})$
$\frac{1}{E(X)}) = \sum p(x)(x)$
For $E(X) = E(\frac{1}{X})$: $\sum(p(x)(x - \frac{1}{x}) = 0$ The above equation might not true be in general. However one possible case where this is true is for this distribution:

$$p(x) = \begin{cases} 1/2 & x = 1 or x = -1 \\ 0 & otherwise \end{cases}$$

2

$a)$ Given there are c objects with j chosen. The probability to select a new 'distinct' component given j are already selected is $\frac{c-j}{c}$. Thus the distribution is geometric with parameter $p = \frac{c-j}{c}$ and the mean being $\frac{1}{p} = \frac{c}{c-j}$
$b)$ Mean time required $= \sum 1 \sum_{j=0}^{c-1} \frac{c}{c-j}$

5

$$f(x) = \begin{cases} \frac{1}{x(x+1)} & x = 1, 2, ... \\ 0 & otherwise \end{cases}$$

$E(X^\alpha) = \sum \frac{x^{\alpha-1}}{x+1}$ For $E(X^\alpha) < \infty$, the above sequence should not be diverging. and hence: $E(X^\alpha) = \sum \frac{1}{x^{2-\alpha}+x^(1-\alpha)}$ and hence $x^{2-\alpha} + x^(1 - \alpha)$ should be converging $\implies \alpha \leq 1$