Math 578B – Fall 2015 – Homework #3

Peter Ralph

due 15 September

1. Let Λ be a Poisson point process on $\mathbb{R}$ with intensity $\lambda(x)$; and mark each point with an independent label from the distribution μ; i.e., take an enumeration of the points $\Lambda = \{x_1, x_2, \ldots\}$, and then let $\Lambda' = \{(x_1, U_1), (x_2, U_2), \ldots\}$, where $U_1, U_2, \ldots$ are i.i.d. random variables from a probability distribution with density μ. Show that Λ' is a Poisson point process on $\mathbb{R}^2$ with intensity $\lambda(x)\mu(y)$.

2. Raindrops fall as a Poisson point process in space ($\mathbb{R}^2$) and time ($[0, \infty)$) with intensity λ drops per cm^2 per second. Each splatters to an independently, randomly chosen radius, with an Exponential(1) distribution. What is the probability density of the radius of the first drop to cover the origin? Check your answer by simulation.

3. Consider the "polymerase complex assembly" chain from the first homework, with transition probabilities

$$P = \begin{array}{c} \\ \varnothing \\ \alpha \\ \beta \\ \alpha+\beta \\ \text{pol} \\ \dagger \end{array} \begin{array}{c} \begin{array}{cccccc} \varnothing & \alpha & \beta & \alpha+\beta & \text{pol} & \dagger \end{array} \\ \left[\begin{array}{cccccc} * & k_\alpha & k_\beta & 0 & 0 & 0 \\ k_\alpha & * & 0 & k_\beta & 0 & 0 \\ k_\beta & 0 & * & k_\alpha & 0 & 0 \\ 0 & k_\beta & k_\alpha & * & k_{\text{pol}} & 0 \\ 0 & 0 & 0 & 0 & * & k_\delta \\ 0 & 0 & 0 & k_\delta & 0 & * \end{array}\right] \end{array}, \tag{1}$$

where the "$*$"s on the diagonal are set so that rows sum to 1. This time, set $k_\alpha = k_\beta = 2\times10^{-6}$, $k_{\text{pol}} = 5\times10^{-6}$, and $k_\delta = 10^{-5}$. Suppose that each step of this discrete-time Markov chain takes 10^{-6} seconds.

a. Write down, and explain how to simulate, the continuous-time Markov chain that approximates this chain (work in units of seconds).

b. Let Y_t denote the continuous-time chain, and define $\tau_\dagger = \inf\{t \geq 0 : Y_t = \dagger\}$. Derive a system of linear equations solved by the mean hitting times $u(a) := \mathbb{E}[\tau_\dagger \mid Y_0 = a]$.

c. Use these to solve for u numerically.

d. Simulate the continuous-time chain and use this to check your answer to *b*.