MATH-578B: Assignment # 2

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Contents

Problem 1

Define $h(w)$ to be an indicator function:

$$h(x) = \begin{cases} 1 & x \in A \\ -1 & x \notin A \end{cases}$$

Now consider $E[h(W)]$:

$$\begin{aligned} E[h(W)] &= P(W \in A) \times 1 + P(W \notin A) \times -1 \\ &= P(W \in A) - (1 - P(W \in A)) \\ &= 2P(W \in A) - 1 \end{aligned} \tag{1.1}$$

Similarly,

$$E[h(Z)] = 2P(Z \in A) - 1 \tag{1.2}$$

where $A \in Z^+$
From (1.1), (1.2)

$$\begin{aligned} E(h(W)) - E(h(Z)) &= 2P(W \in A) - 2P(Z \in A) \\ |E(h(W)) - E(h(Z))| &= 2|P(W \in A) - P(Z \in A)| \\ max_{h:||h||=1}|E(h(W)) - E(h(Z))| &= 2max_{A \in Z^+}|P(W \in A) - P(Z \in A)| \\ &= 2|P(W = 0) - P(Z = 0) + P(W = 1) - P(Z = 1) + \cdots| \\ &= \sum_{k \geq 0} |P(W = k) - P(Z = k)| \end{aligned}$$

Problem 2

$w = 11011011$
Periods: $P(\omega) = \{p \in \{1, 2 \cdots, w-1\} : w_i = w_{i+p}\} \ \forall i \in \{1, 2 \ldots w-p\}$
$w_1 = w_4; w_2 = w_5; w_3 = w_6; \ldots w_5 = w_8$
$w_1 = w_7$
$w_1 = w_8$
Thus

$$P(\omega) = \{3, 6, 7\}$$

$P(\omega) = 6$ can be written as multiple of $P(\omega) = 3$ and hence the principal period

$$P(\omega') = \{3, 7\}$$

The mean $(n-w+1)\mu(w)$ using poisson approximation for the number of clumps is given by: $P(\omega) - \sum_{p \in P(\omega')} P(w^{(p)}w)$
$w = 11011011$
$P(\omega) = \{3, 6, 7\}$
$P(\omega') = \{3, 7\}$

$$\begin{aligned}\mu(w) &= P(11011011) - P(110w) - P(1101101w)\\ &= p^6q^2 - p^2q(p^6q^2) - p^5q^2(p^6q^2)\\ &= p^6q^2(1 - p^2q - p^5q^2)\end{aligned}$$

And hence the mean of the poisson approximation is given by $(n-7)p^6q^2(1-p^2q-p^5q^2)$
where C_j is the even that there are j consecutive occurrences of w starting at that particular location.
$C_j = \{w^{(p_1)}w^{(p_2)} \ldots w^{(p_{j-1}w)}\}$ where p_i is a principal period of w. and the mean of this process is given by $\mu_j(w) = P(C_j) - 2P(C_{j+1}) + P(C_{j+2})$
$P(C_j)$ calculation:
Let there be $kw^{(3)}$ and $j-1-k$ with $w^{(7)}$ among the first $j-1$ occurrences of w starting at i. The probability of this is $\binom{j-1}{k}(p^2q)^k(p^5q^2)^{j-1-k}$
Thus,

$$\begin{aligned}P(C_j) &= P(w)\sum_{k=0}^{j-1}(P(w^{(3)})^k(P(w^{(7)}))^{j-1-k}\\ &= p^6q^2(p^2q + p^5q^2)^{j-1}\\ &= p^{2j+4}q^{j+1}(1+p^3q)^{j-1}\\ P(C1) &= p^6q^2\\ P(C2) &= p^8q^3(1+p^3q)\\ P(C1) - P(C2) &= p^6q^2(1-p^2q-p^5q^2)\\ \mu_j(w) &= p^{2j+4}q^{j+1}(1+p^3q)^{j-1}(1-2p^2q(1+p^3q)+p^4q^2(1+p^3q)^2)\end{aligned}$$

Given a clump, rhe number of occurrences of w can be approximated by:

$$\begin{aligned}P(L_i = j) &= \frac{u_j(w)}{\sum_{i\geq 1}\mu_j(w)}\\ &= \frac{p^{2j+4}q^{j+1}(1+p^3q)^{j-1}(1-2p^2q(1+p^3q)+p^4q^2(1+p^3q)^2)}{p^6q^2(1-p^2q-p^5q^2)}\end{aligned}$$

Exercise 6.1

Expected number of apparent islands = $Ne^{-c(1-\theta)} = cge^{-c(1-\theta)}$
where $g = \frac{G}{L}$ (since $cg \to \infty$) as $g \to \infty$

$$f(c) = cge^{-c(1-\theta)}$$
$$\frac{\partial f(c)}{\partial c} = g(e^{-c(1-\theta)} + -c(1-\theta)e^{-c(1-\theta)}) = 0$$
$$\implies 1 - c(1-\theta) = 0$$
$$c^* = (1-\theta)^{-1}$$

and the maximum number of apparent islands is $Ne^{-c^*(1-\theta)} = Ne^{-1} = \frac{G}{L}e^{-1}(1-\theta)^{-1}$

Exercise 6.5

From our solution for $Excersice 6.1$ we see that the the maximum number of apparent islands occur at $c^* = (1-\theta)^{-1}$ Now c was defined to be the expected number of clones covering a point. and c^* is the maximum number of clones arranged such that no two clones overlap.(overlap if at all is less than θ) Then the maximum number of islands at any point would be $ceil((1-\theta)^{-1}) = 1 + max\{k : \text{k integer}, k < (1-\theta)^{-1}\}$

Excercise 11.14

Stein equation: $(Lf)(x) = \lambda f(x+1) - xf(x)$ and $W = \sum_{i=1}^n X_i$
$E((Lf)(W)) = 0$
And $Z \sim Poisson(E(W))$
Since X_i is iid. Note: $J_i = \{i\}$ i.e since this is an iid scenarios, the only dependent variable to X_i, is X_i itself (this requires p to be $\neq 0$ and $\neq 1$)

$$b_1 = \sum_{i \in I} \sum_{j \in J_i} E(X_i)E(X_j) = \sum_{i=1} p^2 = np^2$$
$$b_2 = \sum_{i \in I} \sum_{i \neq j \in J_i} E(X_i X_j) = \sum_{i} \sum_{i \neq j \in I_i} E(X_i X_j) = 0$$

And hence, by Theorem 11.22:

$$||W - Z|| \leq 2np^2 \frac{1 - e^-\lambda}{\lambda} \leq 2np^2$$

where $EZ = EW = \lambda$

Exercise 11.10

$$X_i = \prod_{j=1}^{i+t-1} D_j$$
$$W = \sum_{i \in I} X_i$$

$I = \{1, 2, \ldots, n-t+1\}$ and $J_i = \{j \in I : |i-j| < t\}$
$EW = \lambda = (n-t+1)p^t$
Since $p < 1$ define $q = 1/p$
Now, $EW = \frac{n-t+1}{q^t}$ and as $n \to \infty$ In order to prevent $EW \to \infty$, we make $t \to n$ when $n \to \infty$

$$i - (t-1) \leq j \leq i + (t-1)$$

$$b_1 = \sum_i p^t \times ((2t-2)+1)p^t = (n-t+1)(2t-1)p^{2t}$$
$$b_2 = \sum_i p^{(2t-2)+1} = (n-t+1)p^{2t-1}$$

Now, Let $q = 1/p$

$$\begin{aligned}\lim_{n\to\infty} b_1 &= \lim_{n\to\infty} \frac{(n-t+1)(2t-1)}{q^{2t}} \\ &= \frac{2(n-t+1)+(-1)(2t-1)}{q^{2t}\ln(q)} \quad \text{Using LHospital Rule} \\ &= \frac{2n-4t+3}{q^{2t}\ln(q)}\end{aligned}$$

Thus, for b_1 to be finite, $t \to n/2$ as $n \to \infty$
and similar is the case for b_2. Thus for $t \to \infty$, b_1, b_2 are both bounded and infact $b_1, b_2 \to 0$ as $n\infty$ or as $t \to n$