MATH-501: Homework # 1

Due on Wednesday, February 11, 2015

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Contents

Problem # 1

1a

$f(x) = atan(x)$ on interval $[a, b] = [-4.9, 5.1]$
$f(a) = atan(-4.9) = 0.7854$ and $f(b) = atan(5.1) = -1.3695$
Since $atan(x) \epsilon C[-4.9, 5.1]$ and $f(-4.9)f(5.1) < 0$ the conditions required for bisection method to converge are satisfied.
The Number of iterations is given by $M = \lceil log_2(\frac{b-a}{2\delta})\rceil$ where δ = Absolute error = 10^-2
Hence $M = \lceil log_2(\frac{10}{2*10^-2})\rceil = 9$

1b

f(-4.9) = -1.369
f(5,1) = 1.377
$c_0 = \frac{a+b}{2} = 0.1$ and $f(c_0) = 0.0997 \implies f(c_0)f(a) < 0$ Hence $c_1 = \frac{c_0+a}{2} = -2.4$ and $f(c_1) = -1.176 \implies f(c_1)f(c_0) < 0$
Hence $c_2 = \frac{c_1+c_0}{2} = -1.15$ and $f(c_2) = -0.855 \implies f(c_2)f(c_0) < 0$ so
$c_3 = \frac{c_2+c_0}{2} = -0.525$ and $f(c_3) = -0.05254$ and $f(c_3)f(c_0) < 0$

1c

ϵ	Number of iteration	Solution	k=$\lceil log_2(\frac{b-a}{2\delta})\rceil$
10^{-2}	9	0.00234375	9
10^{-4}	12	-9.76562500003553e-05	12
10^{-8}	22	9.53674312853536e-08	22
10^{-16}	52	-4.44089209850063e-16	52
10^{-32}	104	-9.86076131526265e-32	104
10^{-64}	212	-3.03858167864314e-64	212
10^{-128}	424	-4.61648930889287e-128	424

2

2a

$x0 = 5$
Iteration : 1 —— x2 = 3.6266 —— x1 = 2.32486
Iteration : 2 —— x2 = 1.16027 —— x1 = 3.6266
Iteration : 3 —— x2 = 2.32486 —— x1 = 1.16027
Iteration : 4 —— x2 = 0.300819 —— x1 = 2.32486
Iteration : 5 —— x2 = 1.16027 —— x1 = 0.300819
Iteration : 6 —— x2 = 0.00861099 —— x1 = 1.16027
Iteration : 7 —— x2 = 0.300819 —— x1 = 0.00861099
Iteration : 8 —— x2 = 2.12823e-07 —— x1 = 0.300819

$x_s = 2.128231491385\,63e - 07$
$g(x_s) = x_s - atan(x_s) = 3.20284333180532e - 21$

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x0 = -5
Iteration : 1 —— x2 = -3.6266 —— x1 = -2.32486
Iteration : 2 —— x2 = -1.16027 —— x1 = -3.6266
Iteration : 3 —— x2 = -2.32486 —— x1 = -1.16027
Iteration : 4 —— x2 = -0.300819 —— x1 = -2.32486
Iteration : 5 —— x2 = -1.16027 —— x1 = -0.300819
Iteration : 6 —— x2 = -0.00861099 —— x1 = -1.16027
Iteration : 7 —— x2 = -0.300819 —— x1 = -0.00861099
Iteration : 8 —— x2 = -2.12823e-07 —— x1 = -0.300819

$x_s = -2.12823149138563e - 07$ $g(x_s) = x_s - atan(x_s) = -3.20284333180532e - 21$

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x0 = 1
Iteration : 1 —— x2 = 0.214602 —— x1 = 0.00320628
Iteration : 2 —— x2 = 1.0987e-08 —— x1 = 0.214602

$x_s = 1.09870240240853e - 08$
$g(x_s) = x_s - atan(x_s) = 0$

$x0 = -1$
Iteration : 1 —— x2 = -0.214602 —— x1 = -0.00320628
Iteration : 2 —— x2 = -1.0987e-08 —— x1 = -0.214602

$x0 = 0.1$
$x_s = 1.21263429527819e - 11$ $g(x_s) = x_s - atan(x_s) = 0$

2b

The number of iterations reduce as the starting value x_0 approaches the exact solution. This is intuitive since we are starting close to $y = x$ and $y = g(x)$ intersection when we start x_0 close to the analytical solution. Hence the number of iterations it would take for the slop to reach $y = x$ is less.

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3a

$|x_{k+1} - \sqrt{a}| \leq \frac{1}{2}|x_k - \sqrt{a}|$

Extending we get.

$|x_{k+1} - \sqrt{a}| \leq \frac{1}{2}|x_k - \sqrt{a}| \leq \frac{1}{4}|x_{k-1} - \sqrt{a}| ... \leq \frac{1}{2^{k+1}}|x_0 - \sqrt{a}|$

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3b

$x0 = 1.1$
Iteration : 1 —— x2 = 1.00455 —— x1 = 1.00001
$x_s = 1.00454545454545$

$x_0 = 2$
Iteration : 1 —— x2 = 1.25 —— x1 = 1.025
Iteration : 2 —— x2 = 1.0003 —— x1 = 1.25
Iteration : 3 —— x2 = 1.025 —— x1 = 1.0003
Iteration : 4 —— x2 = 1 —— x1 = 1.025
$x_s = 1.00000004646115$

$x0 = 5$
Iteration : 1 —— x2 = 2.6 —— x1 = 1.49231
Iteration : 2 —— x2 = 1.08121 —— x1 = 2.6
Iteration : 3 —— x2 = 1.49231 —— x1 = 1.08121
Iteration : 4 —— x2 = 1.00305 —— x1 = 1.49231
Iteration : 5 —— x2 = 1.08121 —— x1 = 1.00305
Iteration : 6 —— x2 = 1 —— x1 = 1.08121
$x_s = 1.00000463565079$

$x0 = 10$
Iteration : 1 —— x2 = 5.05 —— x1 = 2.62401
Iteration : 2 —— x2 = 1.50255 —— x1 = 5.05
Iteration : 3 —— x2 = 2.62401 —— x1 = 1.50255
Iteration : 4 —— x2 = 1.08404 —— x1 = 2.62401
Iteration : 5 —— x2 = 1.50255 —— x1 = 1.08404
Iteration : 6 —— x2 = 1.00326 —— x1 = 1.50255
Iteration : 7 —— x2 = 1.08404 —— x1 = 1.00326
Iteration : 8 —— x2 = 1.00001 —— x1 = 1.08404
$x_s = 1.00000528956427$

$x0 = 50$
Iteration : 1 —— x2 = 25.01 —— x1 = 12.525
Iteration : 2 —— x2 = 6.30242 —— x1 = 25.01
Iteration : 3 —— x2 = 12.525 —— x1 = 6.30242
Iteration : 4 —— x2 = 3.23054 —— x1 = 12.525
Iteration : 5 —— x2 = 6.30242 —— x1 = 3.23054
Iteration : 6 —— x2 = 1.77004 —— x1 = 6.30242
Iteration : 7 —— x2 = 3.23054 —— x1 = 1.77004
Iteration : 8 —— x2 = 1.1675 —— x1 = 3.23054
Iteration : 9 —— x2 = 1.77004 —— x1 = 1.1675
Iteration : 10 —— x2 = 1.01202 —— x1 = 1.77004

$x_s = 1.01201564410353$
The best convergence is obtained in just one iteration when $x_0 = 1.1$ which is self-explanatory since the exact solution is 1. As the value of x_0 moves away from 1, the number of iterations increase too.

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$g(x) = x + cf(x)$
$g'(x) = 1 + cf'(x)$
$g''(x) = cf''(x)$

Sufficent condition for convergence of $x_{n+1} = g(x_n)$: $|g'(x)| < 1$ for x
so $|1 + cf'(x)| < 1$
When $1 + cf'(x) > 0$, $c < \frac{1}{f'(x)}$ or $c < min(\frac{1}{f'(x)})$ for x in domain(f)

5

$g(x) = \frac{1}{3}(\frac{x^3}{3} - x^2 - \frac{5}{4}x + 4)$
$g'(x) = \frac{1}{3}(x^2 - 2x + 1 - \frac{3^2}{2^2})$
$g'(x) = \frac{1}{3}(x - 1 + 1.5)(x - 1 - 1.5)$
$g'(x) = \frac{1}{3}(x - 0.5)(x - 2.5)$ $g''(x) = \frac{1}{3}(2x - 1)$ $g''(x) < 0$ for $x \in [0, 0.5]$ and $g''(x) > 0$ for $x \in [0.5, 2]$
Thus,$g'(x)$ is bounded between $max = g'(0)$ and $min = g'(2)$
$max = g'(0) = 1.25/3$ and $min = -0.5$ and hence $|g'(x)| < 1$ for $x \in [0, 2]$
Thus, by contraction mapping theorem $g(x)$ is convergent(it maps to itself and is continuous)

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$x0 = 0.5$
Iteration : 1 —— x2 = -0.079560 —— x1 = 0.000335
Iteration : 2 —— x2 = -0.000000 —— x1 = -0.079560

$x_s = -2.51314736165673e - 11$

$x0 = 1$
Iteration : 1 —— x2 = -0.570796 —— x1 = 0.116860
Iteration : 2 —— x2 = -0.001061 —— x1 = -0.570796
Iteration : 3 —— x2 = 0.116860 —— x1 = -0.001061
Iteration : 4 —— x2 = 0.000000 —— x1 = 0.116860

$x_s = 7.96309604410642e - 10$

$x0 = 1.3$
Iteration : 1 —— x2 = -1.161621 —— x1 = 0.858896
Iteration : 2 —— x2 = -0.374241 —— x1 = -1.161621
Iteration : 3 —— x2 = 0.858896 —— x1 = -0.374241
Iteration : 4 —— x2 = 0.034019 —— x1 = 0.858896
Iteration : 5 —— x2 = -0.374241 —— x1 = 0.034019
Iteration : 6 —— x2 = -0.000026 —— x1 = -0.374241
Iteration : 7 —— x2 = 0.034019 —— x1 = -0.000026
Iteration : 8 —— x2 = 0.000000 —— x1 = 0.034019

$x_s = 1.20451710400576e - 14$

$x0 = 1.4$
Iteration : 1 —— x2 = -1.413619 —— x1 = 1.450129
Iteration : 2 —— x2 = -1.550626 —— x1 = -1.413619
Iteration : 3 —— x2 = 1.450129 —— x1 = -1.550626
Iteration : 4 —— x2 = 1.847054 —— x1 = 1.450129
Iteration : 5 —— x2 = -1.550626 —— x1 = 1.847054
Iteration : 6 —— x2 = -2.893562 —— x1 = -1.550626
Iteration : 7 —— x2 = 1.847054 —— x1 = -2.893562
Iteration : 8 —— x2 = 8.710326 —— x1 = 1.847054
Iteration : 9 —— x2 = -2.893562 —— x1 = 8.710326
Iteration : 10 —— x2 = -103.249774 —— x1 = -2.893562
Iteration : 11 —— x2 = 8.710326 —— x1 = -103.249774
Iteration : 12 —— x2 = 16540.563827 —— x1 = 8.710326
Iteration : 13 —— x2 = -103.249774 —— x1 = 16540.563827
Iteration : 14 —— x2 = -429721482.896409 —— x1 = -103.249774

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$x0 = 0.5; x1 = 1$
Iteration : 1 —— x2 = -0.220508 —— x1 = -0.220508
Iteration : 2 —— x2 = 0.00922514 —— x1 = -0.220508
Iteration : 3 —— x2 = 0.00922514 —— x1 = 0.00922514

$x_s = 0.00922514366495114$

$x0 = 1; x1 = 1.3$
Iteration : 1 —— x2 = -0.816614 —— x1 = -0.816614
Iteration : 2 —— x2 = 0.0295353 —— x1 = -0.816614
Iteration : 3 —— x2 = 0.0295353 —— x1 = 0.0295353

$x_s = 0.0295353475407389$

$x0 = 1.4; x1 = 1.5$
Iteration : 1 —— x2 = -1.54772 —— x1 = -1.54772
Iteration : 2 —— x2 = -0.0385867 —— x1 = -1.54772
Iteration : 3 —— x2 = -0.0385867 —— x1 = -0.0385867
Iteration : 4 —— x2 = 0.0175067 —— x1 = -0.0385867
Iteration : 5 —— x2 = 0.0175067 —— x1 = 0.0175067
Iteration : 6 —— x2 = -0.00843043 —— x1 = 0.0175067
Iteration : 7 —— x2 = -0.00843043 —— x1 = -0.00843043

$x_s = -0.00843043056177595$

$x0 = 10; x1 = 11$
Iteration : 1 —— x2 = -153.3 —— x1 = -153.3
Iteration : 2 —— x2 = -69.1442 —— x1 = -153.3
Iteration : 3 —— x2 = -69.1443 —— x1 = -69.1442
Iteration : 4 —— x2 = -28.4584 —— x1 = -69.1443
Iteration : 5 —— x2 = -28.4584 —— x1 = -28.4584
Iteration : 6 —— x2 = -8.81641 —— x1 = -28.4584
Iteration : 7 —— x2 = -8.81642 —— x1 = -8.81641
Iteration : 8 —— x2 = 0.549157 —— x1 = -8.81642
Iteration : 9 —— x2 = 0.549156 —— x1 = 0.549157
Iteration : 10 —— x2 = -4.3492 —— x1 = 0.549156
Iteration : 11 —— x2 = -4.34919 —— x1 = -4.3492
Iteration : 12 —— x2 = 2.50353 —— x1 = -4.34919
Iteration : 13 —— x2 = 2.50353 —— x1 = 2.50353
Iteration : 14 —— x2 = -29.3372 —— x1 = 2.50353
Iteration : 15 —— x2 = -29.3373 —— x1 = -29.3372
Iteration : 16 —— x2 = -9.23968 —— x1 = -29.3373
Iteration : 17 —— x2 = -9.2397 —— x1 = -9.23968
Iteration : 18 —— x2 = 0.353472 —— x1 = -9.2397
Iteration : 19 —— x2 = 0.353461 —— x1 = 0.353472
Iteration : 20 —— x2 = -2.54352 —— x1 = 0.353461

$x_s = -2.54352219321986$

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