MATH-501: Homework # 1

Due on Wednesday, February 11, 2015

Saket Choudhary
2170058637

Contents

Problem # 1

1a

$sinx = p_0 + p_1x$ Consider $\|sin(x) - p_1x - p + 0\|\|_2 = \int_{-1}^{1} (sin(x) - p_1x - p_0)^2 dx$
In order to find, p_1, p_0 we consider partial derivatives

$$\frac{d}{dp_1}\int_{-1}^{1} (sin(x) - p_1x - p_0)^2 dx = 0 \tag{1}$$

and

$$\frac{d}{dp_0}\int_{-1}^{1} (sin(x) - p_1x - p_0)^2 dx = 0 \tag{2}$$

Using Liebnitz's formula in 1: $\int_{-1}^{1} 2(-x)(sin(x) - p_1x - p_0)dx = 0 \implies \int_{-1}^{1} x.sin(x) - p_1x^2 - p_0xdx = 0 \implies -x.cos(x) \mid_{-1}^{1} + \int_{-1}^{1} cos(x)dx - \frac{2p1}{3} = 0$ Thus, $p_1 = 3(sin(1) - cos(1))$
Similarly using Leibnitz's ruke on 2: $\int_{-1}^{1} 2(-1)(sin(x) - p_1x - p_0)dx = 0 \implies p_0 = 0$ (The first two terms are odd terms and hence integrate to 0)
p_0 is also justified since $sin(x = 0) = 0$ Hence $sin(x) = 3(sin(1) - cos(1))x$

1b

Taylor approximation(degree 3) around $t = 0$: $p_2(t) = sin(0) + \frac{cos(0)}{1!}(x-0)^1 + \frac{-sin(0)(x-0)^2}{2!} + \frac{-cos(0)(x-0)^3}{3!} + R_4$
$p_2(t) = t - \frac{t^3}{3!} + R_4(t)$ where R_4 is $o(t^4)$ remainer term.

1c

Given $f(t)$ at $t = -1, \frac{-1}{3}, \frac{1}{3}, 1$ we fit a degree3 polynomial for $sin(x)$ using Legendre Polynomials.
$sin(-1) = -sin(1)$ and $sin(\frac{-1}{3}) = -sin(\frac{1}{3})$

Now, $l_0(x) = \frac{(x+\frac{1}{3})(x-\frac{1}{3})(x-1)}{(\frac{-2}{3})(\frac{-4}{3})(-2)} = \frac{-9}{16}(x+\frac{1}{3})(x-\frac{1}{3})(x-1)$

$l_1(x) = \frac{(x+1)(x-\frac{1}{3})(x-1)}{(\frac{2}{3})(\frac{-2}{3})(\frac{-4}{3})} = \frac{27}{16}(x+1)(x-\frac{1}{3})(x-1)$

$l_2(x) = \frac{(x+1)(x+\frac{1}{3})(x-1)}{(\frac{4}{3})(\frac{2}{3}))(\frac{-2}{3})} = \frac{-27}{16}(x+1)(x+\frac{1}{3})(x-1)$

$l_3(x) = \frac{(x+1)(x+\frac{1}{3})(x-\frac{1}{3})}{(2)(\frac{4}{3})(\frac{2}{3}))} = \frac{9}{16}(x+1)(x+\frac{1}{3})(x-\frac{1}{3})$

$sin(x) = \sum_{i=0}^{3} t_i x l_i(x)$

Thus $sin(x) = (-sin(1))l_0(x) + (-sin(\frac{1}{3}))l_1(x) + sin(\frac{1}{3})l_2(x) + sin(1)l_3(x)$

continued on next page...

2

Given: $u_1 = 1, u_2 = x, u_3 = x^2$ $w_i(x) = \frac{v_i}{\|v_i\|}$ where v_i is given by: $v_1 = u_1 = 1$
$v_2 = u_2 - \frac{\langle u_2,v_1\rangle v_1}{\langle v_1,v_1\rangle}$ $v_3 = u_3 - \frac{\langle u_3,v_1\rangle v_1}{\langle v_1,v_1\rangle} - \frac{\langle u_3,v_2\rangle v_2}{\langle v_2,v_2\rangle}$
$v_2 = x - \frac{\int_0^1 xdx}{\int_0^1 1^2 dx} = x - \frac{1}{2}$
$v_3 = x^2 - (\int_0^1 x^2 dx + (x - \frac{1}{2})\frac{\int_0^1 x^2(x-\frac{1}{2})dx}{\int_0^1 (x-\frac{1}{2})^2 dx} = x^2 - x + \frac{1}{6}$
Similarly, $w_1 = 1$
$w_2 = \frac{x-\frac{1}{2}}{\sqrt{\frac{1}{12}}}$
$w_3 = \frac{x^2-x+\frac{1}{6}}{\sqrt{\frac{1}{180}}}$
$Pf = \sum_{i=1}^{3}\langle f, w_i\rangle w_i = (\int_0^1 \sqrt{x}dx)1 + (\int_0^1 \sqrt{x}(x - \frac{1}{2})dx)(x - \frac{1}{2}) + (\int_0^1 \sqrt{x}(x^2 - x + \frac{1}{6})dx)(x^2 - x + \frac{1}{6})$
$Pf = \frac{2}{3} + \frac{4}{5}(x - \frac{1}{2}) + \frac{-4}{7}(x^2 - x + \frac{1}{6})$

3

By Weierstrass' approxmiation theorem for $\epsilon > 0$ there exists a polynomial $p(x)$ such that $\| p - f\|_\infty = max|f(x) - p(x)| < \epsilon$ while $a \leq x \leq b$ $|E_n(f)| = |E_n(f) - E_n(p)|$
$= | \int_0^1 f(x)dx - \sum_{i=1}^n w_i f(x_i)|$
$= | \int_0^1 f(x)dx - \int_0^1 p(x)dx + \sum_{i=1}^n w_i p(x_i) - \sum_{i=1}^n w_i f(x_i)|$
$= | \int_0^1 (f(x) - p(x))dx + \sum_{i=1}^n w_i(p(x_i) - f(x_i))|$
Thus,
$|E_n(f)| = | \int_0^1 (f(x) - p(x))dx + \sum_{i=1}^n w_i(p(x_i) - f(x_i))|$ (Since, $E_n(P) = 0$
Applying triangular inequality,
$E_n(f) \leq= \int_0^1 |(f(x) - p(x))|dx + \sum_{i=1}^n w_i|(p(x_i) - f(x_i))| \approx \| p - f\|_\infty + \| p - f\|_\infty \leq \epsilon$
Thus,

$\| p - f\|_\infty \leq \frac{\epsilon}{2}$ and hence there exists a $N > 0$ such that $|E_n(f)| < \epsilon$ when $n > N$