EE512 Stochastic Processes	**University of Southern California**
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3.1 Stochastic Processes

For each t, $X(t)$ is a random variable

T = range of time

If T is countable we have discrete time stochastic process. If T is interval we have continuous time stochastic process.

Example: Suppose we are tossing a fair coin infinitely many times.

$\Omega = \{$Any infinite string of H and T$\}$

Like $HHHTHTHHTHT$..

Look at range of time interval T = $[1, \infty)$

For $1 \leq tM2$

$$X(t)(\omega) = \begin{cases} 1 & \text{if the first toss is head} \\ -1 & \text{otherwise} \end{cases}$$

For $n \leq tMnH$

$$X(t)(\omega) = \begin{cases} 1 & \text{if the } n^{th} \text{ toss is head} \\ -1 & \text{otherwise} \end{cases}$$

E.g.$X(t)(HTHHTTT) = 1$1 $X(t)(THHTTHHT) = -1$

So we defined collection of random variables $\{X(t), 1 \leq t < \infty\}$ In a stochastic proces, each $X(t)$ is random variable

Fix an ω and look at $X(t)(\omega)$ as a function of time.

Consider $X(t)(\omega)$ for $HHTHHH$....

$$X(t)(w) = \begin{cases} 1 \end{cases}$$

IF we change ω we get a different sample path. So sample path is a realization of the stochastic process. If we fix time we get a random variable

For $n = 1, 2, 3 \dots$

$$Y_n = \begin{cases} 1 & \text{if } n^{th} \text{ toss is heads} \\ 0 & \text{otherwise} \end{cases}$$

Assume coin tosses are independent and $P(heads) = p; P(tails) = q; p + q = 1$

$$\begin{aligned} P(Y_n = 1) &= P(n^{th} \text{ toss is heads} \\ &= p \\ P(Y_n = 0) &= q \end{aligned}$$

$\{Y_n\}_{n=1,2,3}$ are independent and identically distributed random variables.

3.2 Bernoulli Process

$P(Y_1 = 1, Y_2 = 0, Y_3 = 1) = P(Y_1 = 1)P(Y_2 = 0)P(Y_3 = 1) = pqp$

$P(Y_1 + Y_2 + Y_3 = 1) = 3pq^2$

Generalization: $P(\sum Y_i = k) = n\binom{k}{p}^k q^{n-k}$

3.3 Convergence of Random Variables

Suppose $a_a, a_2, \dots a_n$ is seqience of real numbers and $\lim_{n \longrightarrow \infty} a_n = b$ i.e. $\{a_n\}_{n \geq 1}$ converges to b

a_n converges to b if for every $\epsilon > 0$ there exists a natural number N_ϵ such that $|a_n - b| < \epsilon i f n > N_\epsilon$

Almost sure convergence

Convergence with probability 1.

$\{X_n\}_{n=1,2,3}$ are all defined on $(\Omega, \mathcal{F}, P)$

for any $\omega \in \Omega$, $X_1(\omega), X_2(\omega), X_3(\omega) \dots$ Is the sequence of real numbers $\{X_n(\omega)\}$ convergging to number $Y(\omega)$

Look at collection $\{\omega \in \Omega : \{X_n(\omega)\}\text{converges to}Y(\omega)\}$

If $P(\{\omega \in \Omega : \lim_{x \longrightarrow \infty} X_n(\omega) = Y(\omega)\}) = 1$ then we say that r.v $\{X_n\}_{n \geq 1}$ converge to random variable Y almost surely.

Example: $\Omega = (0, 1)$

$X_1(\omega) = 1 \forall \omega in \Omega$ $X_2(\omega) = \begin{cases} 2 & if 0 < w \leq 0.5 \\ 0 & if 0.5 < w < 1 \end{cases}$

$X_n(\omega) = \begin{cases} n & 0 < \omega \le 1/n \\ 0 & \text{otherwise} \end{cases}$

Pick an arbitrary $\omega = 1/3$

$X_1(\omega)X_2(\omega)\dots X_n(\omega) = 123000\dots$

$Y(1/3) = 0$

$\lim_{n\longrightarrow} X(1/3) = Y(1/3)$

For all $\omega \in (0,1)$, $\lim_{n\longrightarrow\infty} X_n(\omega) = Y(\omega)$

So $P(\{\omega : \lim_n X_n(\omega) = Y(\omega)\}) = 1$ so $X(w)$ converges almost surekt

3.3.1 Strong Large of large numbers

$\{X_n\}_n$ are iid. $E[X_n] = \mu < \infty$ and $E[|X_n|] < \infty$

$S_n = X_1 + X_2 + \dots X_n$

Then $\frac{S_n}{n}$ converges almost surely to μ

$P(\{\omega : \lim_n \frac{S_n(\omega}{n} = \mu\}) = 1$

3.3.2 Convergence in probability

We say that $X_n(\omega)$ converges to Y then $P(|X_n - Y_n|) > \epsilon$ converges to 0

Weak law of large numbers

$\{X_n\}$ are iid with $E[X_n] = mu, E[|X_n|]$

$S_n = \sum X_i$ then $\frac{S_n}{n}$ converges tp μ in probability.

Check if $\lim_{n\longrightarrow\infty} P(|\{\frac{S_n}{n}\} - \mu| > \epsilon) = 0$ $E[\frac{S_n}{n}] = \mu$ and $Var(\frac{S_n}{n}) = \frac{\sigma^2}{n}$

$0 \le P(|S_n/n - \epsilon| > \epsilon) \le E[(S_n/n - \mu)^2]/\epsilon^2 = \frac{\sigma^2}{n\epsilon^2}$

Thus, $\lim_{n\longrightarrow\infty} P(|S_n/n - \epsilon| > \epsilon) = 0$

So S_n/n converges to μ in probability.

Thus, almost sure convergence $\implies$ convergence in probability but converse is not true.

Convergence in distribution

$\{X_n\}_{n\ge1}$ We look at cdfs $F_{X_n}(x)$ and we want to check of $\lim_{n\longrightarrow} F_{X_n}(x) = F_Y(x)$ for all x where F_Y is continuous

Example: $Y \sim \mathcal{N}(0,1)$ and $X_n = (-1)^n Y$ Does X_n converge to Y in distrbution

$F_{X_1}(x) = F_{-Y}(x) = P(-Y \leq x) = P(Y \geq -x) = P(Y \leq x) = F_Y(x)$

$F_X(_2)(x) = P(Y \leq x) = F_Y(x)$

Example: Central limit theorem

$\{X_n\}_{n\geq 1}$ are iid random variable with $E[X_n] = \mu$ and $Var[X_n] = \sigma^2$

$Z_n = \frac{\sum_n X_n - n\mu}{\sigma\sqrt{n}}$

$Z_1 = \frac{X_1 - \mu}{\sigma}$

$Z_2 = \frac{X_1 + X_2 - 2\mu}{\sigma\sqrt{2}}$

CLT: $\{Z_n\}_{n\geq 1}$ converges to Y where $Y \sim \mathcal{N}(0,1)$

$\lim_{n\longrightarrow\infty} P(Z_n \leq x) = F_Y(x)$

Almost sure convergence $\implies$ Convergence in probability $\implies$ Convergence in distribution

Convergence in mean square

$\{X_n\}$ converge to Y in mean square of $\lim_{n\longrightarrow\infty} E[(X_n - Y)^2] = 0$

Example: IID X_n and $S_n = \sum X_i$

then $\lim_{n\longrightarrow\infty} E[(\frac{S_n}{n} - \mu)^2] = 0$

i.e. $\lim_{n\longrightarrow\infty} \frac{\sigma^2}{n} = 0$

3.4 How to remember