EE512 Stochastic Processes | **University of Southern California**
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Example: **Branching Processes** Day 0: $\ldots$ (organisms)

Day 1: $Y_1, Y_2,$ (Number of offsprings of i^{th} bacteria)

$\{Y_i\}$ are iid.

$Y_i \in \{0, 2, 4\}$ $p_0 + p_2 + p_4 = 1$

Day n $\implies$ M bacteria ($X_n = m$)

$X_n = \#$ of bacteria on day n

$P(X_{n+1} = j | X_n = M) = P(Y_1 + Y_2 + \cdots + Y_M = j | X_n = M) = P(X_{n+1} = j | X_n = M, X_{n-1} = a, \ldots X_0 = 2)$

so a branching process is markov chain

Multistep Transition Probability

$P(X_{n+m} = j | X_n = 1, X_{n-1}) = P(X_m = j | X_0 = i)$ $m - step$ transition probabilities does on depend on past given the currrent state.

Gambler's ruin: time homegenous $P(X_1 = 1 | X_0 = 0) = P(X_{101} = 1 | X_100 = 0)$

Example: $S = \{1, 2, 3\}$

$$p = \begin{pmatrix} 0.8 & 0.1 & 0.1 \\ 0.1 & 0.8 & 0.1 \\ 0.1 & 0.1 & 0.8 \end{pmatrix}$$

$P(X_2 = 1, X_1 = 3 | X_0 = 2) = P(X_2 + 1 | X_1 = 3) P(X_1 = 3 | X_0 = 2)$

$P(A \cap B) = P(A|B)P(B) \implies P(A \cap B | C) = P(A | B \cap C) P(B|C)$

$P(X_2 = 1, X_1 = 3 | X_0 = 2) = P(X_2 = 1 | X_1 = 3) P(X_1 = 3 | X_0 = 2) = 0.1 * 0.1 = 0.01$

$P(X_2 = 1 | X_0 = 2) = \sum_i P(X_2 = 1, X_1 = i | X_0 = 2) P(X_i = i | X_0 = 2)$

$P(X_2 = j | X_0 = i)$ denoted by $p^2(i, j)$

$p^2(i, j) = \sum_{k \in S} P(X_2 = j, X_1 = k | X_0 = i) = \sum_{k \in S} P(X_2 = j | X_1 = k) P(X_1 = k | X_0 = i) = \sum_{k \in S} p(k, j) p(i, k)$

$p^2 = p.p$ Two steo is product of one step.

So m step transition probabilities are denoted by $p^m(i, j)$ = $P(X_m = j | X_0 = i)$

$m - step$ transition probability matrix is the $m^[th]$ power of one step transition probability matrix.

4.0.1 Champan Kplmogorov equations

$p^{m+n}(i,j) = \sum_{k \in S} p^m(i,k)p^n(k,j)$

$P(X_{m+n} = j | X_0 = i)$

Proof:

$$\begin{aligned} P(X_{n+m} = k | X_0 = i) &= \sum_{k \in S} P(X_{n+m} = j, X_m = k | X_0 = i) \\ &= \sum_{k \in S} P(X_{n+m} = j | X_m = k, X_0 = i) P(X_m = k | X_0 = i) \\ &= \sum_{k \in S} P(X_{n+m} = j | X_m = k) P(X_m = k | X_0 = i) \\ &= \sum_{k \in S} p^n(k,j) p^m(i,k) \end{aligned}$$

Notation

For some event A, $P(A|X_0 = x) = P_x(A)$

For some RV Y, $E[Y|X_0 = x] = E_x[Y]$

Stopping Time

Definition Consider a RV T, takes avalues in the set $\{0, 1, 2, 3\}$. We say that T is a stopping time with respect to the stochastic process. $\{X_n, n \geq 0\}$ if the value of I_N[INCOMPLETE]

$\{X_n, n \geq 0\}$ – Gambler's ruin problem

$T = \min\{k \geq 0 : X_k \geq 5\}$

Example: $X_0 = 10$ then $T = 0$

Calculate $I_{\{T_n\}}$ from $X_0, X_1, \ldots X_n$

If $\{T = 0\} \implies \{X_0 \geq 5\}$

$\{T = 1\} \implies \{X_0 < 5, X_1 \geq 5\}$

$\{T = n\} \implies \{X_0 < 5, X_1 < 5, \ldots X_{n-1} < 5, X_n \geq 5\}$

T is a stopping time

Strong markov property:

Suppose T is a stopping time with respect to the Markov Chain, $\{X_n, n \geq 0\}$ Given that stopping time $T = n$ and $X_T = y$

Any other information about $X_0, X_1, \ldots X_{T-1}$ is irrelevant for predicting future.

$P(X_{T+1} = j|T = n, X_T = y) = P(X_1 = j|X_0 = y) = p(y, j)$

$P(X_{T+1} = j|T_n, X_T = y, X_{T-1} = x, \dots X_o = z) = P(X_{T+1} = j|T = n, X_T = y)P(y, j)$

$P(X_{T+m} = j|T = n, X_T = y, X_{T-1} = x) = P(X_{T+m} = j|T = n, X_T = y) = P(X_m = j|X_0 = y) = p^m(y, j)$