EE512 Stochastic Processes | **University of Southern California**
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NOTE: Durrett's book for Markov Chain.

Stopping time

T is a stopping time wrt $\{X_n\}_{n\geq 0}$ if $\{T = n\}$ can be determined from $X_0, X_1, \ldots X_n$

Strong markov property

$P(X_{T+m} = z|X_Y = y, T = n) = P(X_m = z|X_0 = y) = p^m(z)$

Time of first return

State space of MC is finite.

For a state $y \in S$, $T_y = \min\{n \geq 1 : X_n = y\}$

since $n \geq 1$ so X_0 is not relevant here.

If $X_n \neq y$ for any finite $n \geq 1$, then we say that $T_y = \infty$

Is T_y a stopping time?

For $I_{T_y=1}$ all we need to know is X_1 $[= y, \neq y]$

If $\{T_y = k\}$ then $X_1, X_2, X_{k-1} \neq y$ and $X_k = y$

$P_y(T_y < \infty) = P(T_y < \infty|X_0 = y)$

Time to return to y is finite given start $X_0 = y$.

$P(T_y < \infty|X_0 = y) = \sum_{n\geq 1} P(T_y = n|X_0 = y) = \rho_{yy}$

ρ_{yy}: Probability of 1st return to y happening in finite time given $X_0 = y$

Time of 2^{nd} return

$T_y^2 = \min\{n > T_1 : X_n = y|X_0 = y\}$

$P(T_y^2 < \infty) =?$

$\{T_y^2 M\infty\} \subset \{T_y < \infty\}$

Thus, $P_y(T_y^2 < \infty) \leq \rho_{yy}$

$P(T_y^2 < \infty) = P(T_y^2 < \infty, T_y^1 < \infty) + P(T_y^2 < \infty, T_y^1 = \infty) = P(T_y^2 < \infty, T_y^1 < \infty) = P(Y_y^2 < \infty|T_y^1 < \infty)P(T_y^1 < \infty) = \rho_{yy}^2$

Inductively, $P(T_y^k < \infty) = \rho_{yy}^k$

Two cases:

- $\rho_{yy} = 1$ In this case we say that y is a RS[Recurrent State]. $P_y(T_y^1 < \infty) = 1$ so guaranteed to come back to y $P_y(T_y^k < \infty) = 1$. Number of times the MC visits given $X_0 = y$ is infinite
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Let $N(y)$ = Total number of visits to y from time 1 onwards $\{N(y) = \infty\} = \cup_{k \geq 0}\{N(y) \geq k\}$

$P_y(N(y) = \infty) = P(\cup_{k\geq 0}\{N(y) \geq k\}) = \lim_{k \longrightarrow \infty} P_y(N(y) > k) = \lim_{k\longrightarrow\infty} P(T_y^k < \infty) = \lim_{k\longrightarrow\infty} \rho_{yy}^k = 1$

IF $\rho_{yy} < 1$ then $P(T_y' < \infty) < 1$ and $P(T_y' = \infty) > 0$

States with $\rho_{yy} < 1$ are called transient state

$$\begin{aligned} P(N(y) = \infty) &= P_y(\cap\{N(y) \geq k\} \\ &= \lim_{k \longrightarrow \infty} P(T_y^k < infty) \\ &= \rho_{yy}^k \\ &= 0 \end{aligned}$$

For recurrent states: $N_y = \infty$ with probability 1. for transient states, $N_y < \infty$ with probability 1.

Example

$\rho_{00} = 1$ so state 0 is recurrent[also absorbing]

In general any absorbing state is also recurrent.

Simimarly state 4 is also recurrant.

If we start at 1, we visit 1 only finitel maaaaaaaaa Recurring states do not have to be absorbing states.

Example 2 A-A : 0.5 A:B : 0.5 B-B : 1

Both $0, 1$ are recurrent

$P(T_0^1 = k) = \frac{1}{2^k}$

$P(T_0^1 < \infty) = 1$ Thus, 0 is a recurring state, but not an absorbing state.

Recurrent and Transient state

$\rho_{xy} = P(T_y < \infty | X_0 = x) = P_x(T_y < \infty)$

if $\rho_{xy} = 0 \implies P_x(T_y = \infty) = 1$

E.g A-A = 1 B-A = 0.6 B-C=0.4 C-C=1

$\rho_{10} = 0.6$

Suppose there exists a finite number m such that the m step transition probab from x to y is positivie

$p^m(x, y) > 0$, then $\rho_{xy} > 0$

Converse: Suppose $\rho_{xy} > 0$ then there exists a finite number m such that $p^m(x, y) > 0$

$\rho_{xy} = P_x(T_y < \infty) = \sum_{m=1} P_x(T_y = m) > 0 \implies$ there exists at least one $m = n$ such that $P_x(T_y = n) > 0$

$P_x(T_y = n) \leq P_x(X_n = y) \implies p^n(xy) < 0$

If $\rho_{xy} > 0$ then we say that x, y communicate or $x \longrightarrow y$

Theorem If $\rho_{xy} > 0$ and $\rho_{yx} < 1$ then x is a transient state.

Example:

$\rho_{21} = 1$

$\rho_{12} = 0$

So 2 is transient