ISE 538 Homework 1

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Problem 1.31

Let D_i represent the score on i^{th} dice. Then the possible configurations for getting a sum 7 are: (D_1, D_2) : $(1,6),(2,5),(3,4),(4,3),(5,2),(6,1)$

Thus $P(D_1 = 6|D_1 + D_2 = 7) = \frac{1}{6}$

Problem 1.32

Let

$$I_i = \begin{cases} 1 & i^{th} \text{ person gets his hat back} \\ 0 & \text{otherwise} \end{cases}$$

$$\begin{aligned}
P(\text{no person gets his hat back}) &= 1 - P(I_1 \cup I_2 \dots \cup I_n) \\
P(I_i = 1) &= \frac{1}{n} \\
P(I_i = 1, I_j = 1) &= \frac{1}{n}\frac{1}{n-1} = \frac{1}{n(n-1)} = \frac{(n-2)!}{n!} \\
P(I_i = 1, I_j = 1) &= \frac{1}{n}\frac{1}{n-1}\frac{1}{n-2} = \frac{(n-3)!}{n!} \\
P(I_1 \cup I_2 \dots \cup I_n) &= \sum_{i=1}^{n} P(I_i) - \sum_{i<j} P(I_iI_j) + \sum_{i<j<k} P(I_iI_jI_k) - \dots - (-1)^n P(I_1I_2 \dots I_n)
\end{aligned}$$

The number of terms of type $P(I_iI_j)$ for $i < j$ are $\binom{n}{2}$, similarly of type $P(I_iI_jI_k)$ for $i < j < k$ are $\binom{n}{3}$ and so on

Thus,

$$\begin{aligned}
P(I_1 \cup I_2 \dots \cup I_n) &= n \times \frac{1}{n} - \binom{n}{2} \times \frac{(n-2)!}{n!} + \binom{n}{3} \times \frac{(n-3)!}{n!} - \dots - (-1)^n \binom{n}{n} \times \frac{1}{n!} \\
&= 1 - \frac{1}{2!} + \frac{1}{3!} - \dots - (-1)^n \frac{1}{n!} \\
P(\text{no person gets his hat back}) &= 1 - (1 - \frac{1}{2!} + \frac{1}{3!} - \dots - (-1)^n \frac{1}{n!}) \\
&= \frac{1}{2!} - \frac{1}{3!} + \dots + (-1)^n \frac{1}{n!}
\end{aligned}$$

Problem 1.34

Assuming that the red and black are equiprobable, choosing a red or black is equally rewarding. $P(11 \text{consecutive black}) = (\frac{1}{2})^{11}$ $P(11^{th}\text{red, last 10 black}) = \frac{1}{2}\frac{1}{2}^{10}$. The probability of observing 10 blacks continuously is very low, having observed this it is actually likely that the system is biased towards black hits and hence the bet should probably be on a black rather than a red.

Problem 1.35

(a) H,H,H,H: $(\frac{1}{2})^4$

(b) T,H,H,H: $(\frac{1}{2})^4$

(c) T,H,H,H occurs before H,H,H,H: The only possible way for H, H, H, H to occur first is that first four flips are heads. And hence the required probability is $1 - \frac{1}{2}^4 = \frac{15}{16}$

Problem 1.36

Box1 : $1B, 1W$

Box2: $2B, 1W$

$P(B) = \frac{1}{2} \times \frac{1}{2} + \frac{1}{2} \times \frac{2}{3} = \frac{7}{12}$

Problem 1.37

$P(Box_1|W) = \frac{P(W,Box1)}{P(W)} = \frac{P(W|Box1)P(Box1)}{P(W)} = \frac{\frac{1}{2}\frac{1}{2}}{1-\frac{7}{12}} = \frac{3}{5}$

Problem 1.44

$P(tails|W) = \frac{P(tails)P(W|tails)}{P(W)}$

$P(W) = P(W|heads)P(heads) + P(W|tails)P(tails) = \frac{5}{12}\frac{1}{2} + \frac{3}{15}\frac{1}{2} = \frac{37}{120}$

Thus, $P(tails|W) = \frac{\frac{1}{2}\frac{3}{15}}{\frac{37}{120}} = \frac{12}{37}$

Problem 1.45

$P(B1|R2) = \frac{P(R2|B1)P(B1)}{P(R2)}$

$$
\begin{aligned}
P(B1) &= \frac{b}{b+r} \\
P(R2) &= P(R2|B1)P(B1) + P(R2|R1)P(R1) \\
&= \frac{r}{r+b+c}\frac{b}{b+r} + \frac{r+c}{r+b+c}\frac{r}{b+r} \\
&= \frac{br + r^2 + rc}{(b+r)(b+r+c)} \\
P(B1|R2) &= \frac{\frac{r}{r+b+c}\frac{b}{b+r}}{\frac{br+r^2+rc}{(b+r)(b+r+c)}} \\
&= \frac{br}{br + r^2 + rc} \\
&= \frac{b}{b+r+c}
\end{aligned}
$$

Problem 1.46

Let's assume WLOG, that the jailer tells C will be set free. Then $P(\text{C free}|\text{A dies}) = \frac{1}{2}$ because is A is to die the jailer could have also named B with equal probability.

$$\begin{aligned}
P(\text{A dies}|\text{C free}) &= \frac{P(\text{C free}|\text{A dies})P(\text{A dies})}{P(\text{C free})} \\
&= \frac{P(\text{C free}|\text{A dies})P(\text{A dies})}{P(\text{C free}|\text{A dies})P(\text{A dies}) + P(\text{C free}|\text{B dies})P(\text{B dies}) + P(\text{C free}|\text{C dies})P(\text{C dies})} \\
&= \frac{\frac{1}{2}\frac{1}{3}}{1\frac{1}{3} + 1\frac{1}{3} + 0} \\
&= \frac{1}{3}
\end{aligned}$$

So the jailer is wrong. The probability of A being executed remains the same.

Problem 1.47

$$0 \leq P(A|B) \leq 1$$

$$P(\Omega|B) = \frac{P(\Omega \cap B)}{P(B)} = \frac{P(B)}{P(B)} = 1$$

For the third property, let's consider disjoint events A and C then

$$\begin{aligned}
P(A \cup C|B) &= \frac{P((A \cup C)B)}{P(B)} \\
&= \frac{P(AB \cup CB)}{P(B)} \\
&= \frac{P(AB) + P(CB)}{P(B)} \\
&= P(A|B) + P(C|B)
\end{aligned}$$

Now,

$$\begin{aligned}
P(A|BC)P(C|B) + P(A|BC^c)P(C^c|B) &= \frac{P(ABC)}{P(BC)}\frac{P(BC)}{P(B)} + \frac{P(ABC^c)}{P(BC^c)}\frac{P(BC^c)}{P(B)} \\
&= \frac{P(ABC)}{P(B)} + \frac{P(ABC^c)}{P(B)} \\
&= P(AC|B) + P(AC^c|B) \\
&= P(AC \cup AC^c|B) \\
&= P(A|B)
\end{aligned}$$