MATH 505B Homework 4

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Problem 6.14.1

$$
\begin{aligned}
\langle \mathbf{x}, \mathbf{Py} \rangle &= \sum_{k\in\theta} x_k (\mathbf{Py})_k \pi_k \\
&= \sum_{k\in\theta} x_k (\sum_j p_{kj} y_j) \pi_k \\
&= \sum_{k\in\theta} x_k (\sum_j p_{kj} \pi_k y_j) \\
&= \sum_{k,j} x_k p_{kj} \pi_k y_j \\
&= \sum_{k,j} x_k (p_{jk} \pi_j y_j) \text{ using reversibility criterion } \pi_j p_{jk} = \pi_k p_{kj} \\
&= \sum_j p_{jk} x_k \pi_j y_j \\
&= \sum_{j\in\theta} (\sum_k p_{jk} x_k) \pi_j y_j \\
&= \langle \mathbf{Px}, \mathbf{y} \rangle
\end{aligned}
$$

Problem 6.14.2

For reversibility: $\pi_i p_{ij} = \pi_j p_{ji}$:

$p'_{ij} = b_{ij} g_{ij} = \frac{\pi_j g_{ji} g_{ij}}{\pi_i g_{ij} + \pi_j g_{ji}}$ and $p'_{ji} = \frac{\pi_i g_{ij} g_{ij}}{\pi_i g_{ij} + \pi_j g_{ji}}$

Hence, $\pi_i p'_{ij} = \pi_j p'_{ji}$ and hence b_{ij} satisfies reversibility criterion besides $0 \leq b_{ij} \leq 1$.

Problem 7.2.1

7.2.1(a)

$$
\begin{aligned}
\{E(|X+Y|^p)\}^{1/p} &= \{E|X^p|\}^{1/p} + \{E|Y^p|^{1/p}\} \\
E[|X|] &= E[|X_n + X - X_n|] \\
\{E[|X|^p]\}^{1/p} &= \{E[|X_n + X - X_n|^p]\}^{1/p} \\
&\leq \{E[|X_n|^p]\}^{1/p} + \{E[|X - X_n|^p]\}^{1/p} \\
\implies \lim_{n \longrightarrow \infty} \inf E[|X|^p] &\leq E[|X_n|^p]^{1/p} \qquad (1)
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
\{E[|X_n|^p]\}^{1/p} &= \{E[|X_n - X + X|^p]\}^{1/p} \\
&\leq \{E[|X|^p]\}^{1/p} + \{E[|X_n - X|]\}^{1/p} \\
\implies \lim_{n \longrightarrow \infty} \sup E[|X_n|^p] &\leq E[|X|^p] \qquad (2)
\end{aligned}
$$

Combining 1, 2 : $E[|X_n|^p] \longrightarrow E[|X|^p]$ $p \geq 1$

7.2.1(b)

Using $p = 1$ in part (a)

7.2.1(c)

Using part (a) $E[X_n^2] \longrightarrow E[X^2]$ $(X_n \xrightarrow{2} X) \Rightarrow (X_n \xrightarrow{1} X) \implies E[X_n] \longrightarrow E[X]$ and hence $Var(X_n) \longrightarrow Var(X)$

Problem 7.2.3

Consider $k \geq 0$, $n \geq 1$, $X_n = k/n \leq X < (k+1)/n$. $X - 1/n \leq X_n \leq X$

Define similarly Y_n. Y_n, X_n are independent by definition

$E[X_n] \longrightarrow E[X]$ and $E[Y_n] \longrightarrow E[Y]$

Thus, using independence and convergence relations $E[X_nY_n] = E[X_n]E[Y_n] \longrightarrow E[X]E[Y]$

Now,

$$
\begin{aligned}
(X - 1/n)(Y - 1/n) &\leq X_nY_n \leq XY \\
\implies E[(X - 1/n)(Y - 1/n)] &\leq E[X_nY_n] \leq E[XY]
\end{aligned}
$$

$$
E[(X - 1/n)(Y - 1/n)] = E[XY - \frac{X+Y}{n} + 1/n^2] \implies E[X_nY_n] \longrightarrow E[XY]
$$

Thus combining, the above two results $E[X_nY_n] \longrightarrow E[XY]$ and $E[X_nY_n] \longrightarrow E[X][Y]$ we get $E[XY] = E[X]E[Y]$

Problem 7.2.10

$\sum_r X_r \sim Poisson(\sum_r \lambda_r)$
Define $t = \sum_{r=1}^n \lambda_r$:

$$P(\sum_r X_r \leq x) = \sum_{i=0}^{x} \frac{e^{-t}t^i}{i!}$$

$$\lim_{n \longrightarrow \infty} P(\sum_r X_r \leq x) = \begin{cases} 0 & t \longrightarrow \infty \\ Poisson(t) & t \text{is finite} \end{cases}$$

Problem 7.4.1

$$\begin{aligned} E[X_1] &= 0 * (1 - \frac{1}{n \log n} + 0 * \frac{1}{2n \log n} \\ &= 0 \\ E[X_1^2] &= \frac{2n^2}{2n \log n} + 0 * (1 - \frac{1}{n \log n} \\ &= \frac{n}{\log n} \\ E[(\frac{1}{n}S_n - 0)^2] &= \frac{1}{n^2} Var(S_n) \\ &= \frac{1}{n^2} \frac{n}{\log n} \\ &= \frac{1}{n \log n} & \longrightarrow 0 \end{aligned}$$

$\sum_i P(|X_i| \geq i) \longrightarrow \infty$ Hence, using Borel-Cantelli Lemma(7.3.10b) we have $P(|X_j| \geq j) = 1$ for some j $|X_j| = |S_j - S_{j-1}| \geq j$ and hence S_j/j diverges.

Problem 7.5.1

Define $I_i(j)$ as the indicator variable denoting if the X_j lies in the i^{th} interval,

$$\begin{aligned} \log R_m &= \sum_{i=1}^{n} Z_m(i) \log p_i \\ &= \sum_{i=1}^{n} \sum_{j=1}^{m} I_i(j) p_i \end{aligned}$$

Define $\sum_{i=1}^m I_i(j) = Y_j$, then $\log R_m = \sum_{j=1}^m Y_j$
$E[Y_j] = \sum_{i=1}^n p_i \log p_i = -h$ Thus, by strong law of convergence $\frac{1}{m} \sum_{j=1}^m Y_j = \longrightarrow -h = E[Y_j]$

Problem 7.5.3

Transient $P(X_n = i|X_0 = i) < 1$

Using strong law $S_n/n \longrightarrow E[X_1]$ If $E[X_1] \neq 0$ then $P[S_n = 0|S_1 = 0] < 1$ as $S_n = 0$ happens only finitely often

Problem 7.7.1

$$\begin{aligned}
E[X_iX_j] &= E[E[X_iX_j|X_0, X_1, \ldots, X_{j-1}]] \\
&= E[E[X_i(S_j - S_{j-1})|X_0, X_1, \ldots, X_{j-1}]] \\
&= E[X_i(E[S_j - S_{j-1}|X_0, X_1, \ldots, X_{j-1}])] \\
&= E[X_i(E[S_j|X_0, X_1, \ldots, X_{j-1}] - S_{j-1})] \\
&= E[X_i(S_{j-1} - S_{j-1})] \\
&= 0
\end{aligned}$$

Problem 7.7.3

$$\begin{aligned}
E[X_{n+1}|X_0, X_1, \ldots, X_n] &= aX_n + X_{n-1} \\
E[S_{n+1}|X_0, X_1, \ldots, X_n] &= E[\alpha X_{n+1} + X_n|X_0, X_1, \ldots, X_n] \\
&= \alpha E[X_{n+1}|X_0, X_1, \ldots, X_n] + X_n \\
&= (\alpha a + 1)X_n + \alpha b X_{n-1} \\
&= S_n = \alpha X_n + X_{n-1} \\
\implies \alpha = \frac{1}{1-a}, b = \frac{1}{\alpha}
\end{aligned}$$

Problem 7.7.4

X_n: Net profit per unit stake on n^{th} play.

$S_i = S_{i-1} + f_i(X_1, X_2, \ldots, X_i)$ such that $S_1 = X_1Y$

Thus, $S_n = \sum_{i=1}^n X_i f_{i-1}(X_1, X_2, \ldots, X_{i-1})$

$$\begin{aligned}
S_{n+1} &= S_n + f_{n+1}(X_1, X_2, \ldots, X_n)X_{n+1} \\
E[S_{n+1} - S_n|X_1, X_2, \ldots, X_n] &= E[X_{n+1}f_{n+1}(X_1, X_2, \ldots, X_n)|X_1, X_2, \ldots, X_n] \\
&= f_{n+1}(X_1, X_2, \ldots, X_n)E[X_{n+1}|X_1, X_2, \ldots, X_n] \\
&= 0 \\
\implies E[S_{n+1}|X_1, X_2, \ldots, X_n] = S_n
\end{aligned}$$

Problem 7.8.1

$E[X_i] = 0$ By Doob-Kolmogorov inquality:

$$
\begin{aligned}
P(\max_{i \le j \le n} |S_j| > \epsilon) &\le \frac{1}{\epsilon^2} \sum_{j=1}^{n} E[S_n^2] \\
E[S_n^2] &= Var(S_n) + E[S_n]^2 \\
&= Var(S_n) \\
&= \sum Var(X_i)
\end{aligned}
$$

$$
\implies P(\max_{i \le j \le n} |S_j| > \epsilon) \le \frac{1}{\epsilon^2} \sum_{j=1}^{n} Var(X_j)
$$

Problem 7.8.3

By theorem 7.8.1 S_n converges to S almost surely Now, using the above proved fact that $S_n \longrightarrow S \implies Var(S_n) \longrightarrow Var(S) \implies Var(S) \longrightarrow 0$