MATH 542 Final Exam

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May 6, 2016

Problem 1

Problem 1a

$$
\begin{aligned}
Y_i &= \beta_0 + \beta_1 x_{i1} + \beta_2 x_{i2} + \cdots + \beta_{p-1} x_{i,p-1} + \epsilon_i \\
Y_i &= \alpha + \beta_1(x_{i1} - \bar{x}_1) + \beta_2(x_{i2} - \bar{x}_2) + \cdots + \beta_{p-1}(x_{i,p-1} - \bar{x}_{p-1}) \\
&= \alpha - \beta_1\bar{x}_1 - \beta_2\bar{x}_2 - \cdots - \beta_{p-1}\bar{x}_{p-1} + \beta_1 x_{i1} + \beta_2 x_{i2} + \cdots + \beta_{p-1} x_{i,p-1} + \epsilon_i \\
\implies \beta_0 &= \alpha - \beta_1\bar{x}_1 - \beta_2\bar{x}_2 - \cdots - \beta_{p-1}\bar{x}_{p-1} \\
\implies \alpha &= \beta_0 + \beta_1\bar{x}_1 - \beta_2\bar{x}_2 - \cdots - \beta_{p-1}\bar{x}_{p-1}
\end{aligned}
$$

Problem 1.2

$$
\begin{aligned}
\bar{x_j} &= \sum_{i=1}^{n} \frac{x_{ij}}{n} \\
&= \frac{1}{n}\mathbf{1_n}'\mathbf{X_i} \\
Y &= \begin{pmatrix} 1 & x_{11}-\bar{x}_1 & x_{12}-\bar{x}_2 & \dots & x_{1,p-1}-\bar{x}_{p-1} \\ 1 & x_{21}-\bar{x}_2 & x_{22}-\bar{x}_2 & \dots & x_{2,p-1}-\bar{x}_{p-1} \\ \vdots & & & & \\ 1 & x_{n1}-\bar{x}_1 & x_{n2}-\bar{x}_2 & \dots & x_{n,p-1}-\bar{x}_{p-1} \end{pmatrix} \begin{pmatrix} \alpha \\ \beta_1 \\ \vdots \\ \beta_{p-1} \end{pmatrix} + \epsilon \\
&= (\begin{pmatrix} 1 & x_{11} & x_{12} & \dots & x_{1,p-1} \\ 1 & x_{21} & x_{22} & \dots & x_{2,p-1} \\ \vdots & & & & \\ 1 & x_{n1} & x_{n2} & \dots & x_{n,p-1} \end{pmatrix} - \begin{pmatrix} 0 & \bar{x}_1 & \bar{x}_2 & \dots & \bar{x}_{p-1} \\ 0 & \bar{x}_1 & \bar{x}_2 & \dots & \bar{x}_{p-1} \\ \vdots & & & & \\ 0 & \bar{x}_1 & \bar{x}_2 & \dots & \bar{x}_{p-1} \end{pmatrix}) \begin{pmatrix} \alpha \\ \beta_1 \\ \vdots \\ \beta_{p-1} \end{pmatrix} + \epsilon \\
&= \left(\begin{pmatrix} \mathbf{1_n} & \mathbf{X} \end{pmatrix} - \begin{pmatrix} \mathbf{0_n} & \frac{1}{n}\mathbf{1_n}\mathbf{1_n'}\mathbf{X} \end{pmatrix} \right) \begin{pmatrix} \alpha \\ \beta_1 \\ \vdots \\ \beta_{p-1} \end{pmatrix} + \epsilon \\
&= \begin{pmatrix} \mathbf{1_n} & \mathbf{X} - \frac{1}{n}\mathbf{1_n}\mathbf{1_n'}\mathbf{X} \end{pmatrix} \begin{pmatrix} \alpha \\ \beta_1 \\ \vdots \\ \beta_{p-1} \end{pmatrix} + \epsilon \\
&= \begin{pmatrix} \mathbf{1_n} & \mathbf{X_c} \end{pmatrix} \begin{pmatrix} \alpha \\ \beta_1 \\ \vdots \\ \beta_{p-1} \end{pmatrix} + \epsilon \\
\implies \mathbf{X_c} &= (\mathbf{I} - \frac{\mathbf{1_n}\mathbf{1_n'}}{n})X
\end{aligned}
$$

Problem 1.3

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\begin{aligned}
\mathbf{1_n'}\mathbf{X_c} &= \begin{pmatrix} 1 & 1 & 1 & \dots & 1 \end{pmatrix} \left(\mathbf{I} - \frac{1}{n}\mathbf{1_n}\mathbf{1_n'} \right) \mathbf{X} \\
&= \begin{pmatrix} 1 & 1 & 1 & \dots & 1 \end{pmatrix} \begin{pmatrix} 1-\frac{1}{n} & \frac{-1}{n} & \frac{-1}{n} & \dots & \frac{-1}{n} \\ \frac{-1}{n} & 1-\frac{1}{n} & -\frac{1}{n} & \dots & -\frac{1}{n} \\ \vdots & & & & \\ \frac{-1}{n} & -\frac{1}{n} & -\frac{1}{n} & \dots & 1-\frac{1}{n} \end{pmatrix} \\
&= (1-\frac{1}{n}) - \frac{1}{n} * (n-1) \\
&= 0
\end{aligned}
$$

Problem 1.4.a

$$
\begin{aligned}
Y_1 &= \alpha + \beta_1(x_{11} - \bar{x}_1) + \beta_2(x_{12} - \bar{x}_2) + \cdots + \beta_{p-1}(x_{1,p-1} - \bar{x}_{p-1}) \\
Y_2 &= \alpha + \beta_1(x_{21} - \bar{x}_1) + \beta_2(x_{22} - \bar{x}_2) + \cdots + \beta_{p-1}(x_{2,p-1} - \bar{x}_{p-1}) \\
&\vdots \\
Y_n &= \alpha + \beta_1(x_{n1} - \bar{x}_1) + \beta_2(x_{n2} - \bar{x}_2) + \cdots + \beta_{p-1}(x_{n,p-1} - \bar{x}_{p-1})
\end{aligned}
$$

$$
\begin{aligned}
\sum_{i=1} Y_i &= n\alpha + \beta_1(\sum_{i=1} x_{i1} - n\bar{x}_1) + \beta_2(\sum_{i=1} x_{i2} - n\bar{x}_2) + \cdots + \beta_{p-1}(\sum_{i=1} x_{i,p-1} - n\bar{x}_{p-1}) \\
\sum_{i=1} x_{i,j} - n\bar{x}_j &= 0 \\
\implies \hat{\alpha} &= \frac{\sum_{i=1} Y_i}{n}
\end{aligned}
$$

Now, we perform $Y_i - \bar{Y}$ eliminating α Let $Z = Y_i - \bar{Y}$m the problem then reduces to the following form: $Z = \mathbf{X_c}\beta + \epsilon$ where $\beta = \begin{pmatrix}\beta_1 & \beta_2 & \dots & \beta_{p-1}\end{pmatrix}$ and hence simply re-using OLS results, $\hat{\beta} = \mathbf{X_c'X_c}^{-1}\mathbf{X_c'}Y$

More rigorously:

$$
\begin{aligned}
\hat{\beta} &= (X'X)^{-1}X'Y \\
X'X\hat{\beta} &= X'Y \\
\mathbf{X'X} &= \begin{pmatrix}\mathbf{1_n}' \\ \mathbf{X_c}'\end{pmatrix}\begin{pmatrix}\mathbf{1_n} & \mathbf{X_c}\end{pmatrix} \\
&= \begin{pmatrix}\mathbf{1_n}'\mathbf{1_n} & \mathbf{1_n'X_c} \\ \mathbf{X_c'1_n} & \mathbf{X_c'X_c}\end{pmatrix} \\
\begin{pmatrix}\mathbf{1_n}'\mathbf{1_n} & \mathbf{1_n'X_c} \\ \mathbf{X_c'1_n} & \mathbf{X_c'X_c}\end{pmatrix}\begin{pmatrix}\alpha \\ \beta_1\end{pmatrix} &= \begin{pmatrix}\mathbf{1_n'} \\ \mathbf{X_c'}\end{pmatrix}\mathbf{Y} \\
\implies n\hat{\alpha} + \mathbf{1_n'X_c}\beta_1 &= \sum_{i=1} Y_i = n\bar{Y}\text{using first row} \\
\hat{\alpha} + 0 &= \bar{Y} \text{ since } \mathbf{X_c'1_n1} = 0 \\
\implies \hat{\alpha} &= \bar{Y} \\
\mathbf{X_c'1_n1_n'} + \mathbf{X_c'X_c}\hat{\beta}_1 &= \mathbf{X_c'Y}\text{using second row} \\
\implies \mathbf{X_c'X_c}\hat{\beta}_1 &= \mathbf{X_c'Y} \text{ since } \mathbf{X_c'1_n1} = 0 \\
\implies \hat{\beta}_1 &= (\mathbf{X_c'X_c})^{\mathbf{-1}}\mathbf{X_c'Y}
\end{aligned}
$$

Problem 1.4.b

Inverse of a block matrix

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\begin{pmatrix}A_{11} & A_{12} \\ A_{21} & A_{22}\end{pmatrix}^{-1} = \begin{pmatrix}(A_{11} - A_{12}A_{22}^{-1}A_{21})^{-1} & -(A_{11} - A_{12}A_{22}^{-1}A_{21})^{-1}A_{12}A_{22}^{-1} \\ -(A_{22} - A_{21}A_{11}^{-1}A_{12})^{-1} & (A_{22} - A_{21}A_{11}^{-1}A_{12})\end{pmatrix}
$$

$$
\begin{aligned}
\mathbf{X'X} &= \begin{pmatrix} \mathbf{1_n}' \\ \mathbf{X_c}' \end{pmatrix} \begin{pmatrix} \mathbf{1_n} & \mathbf{X_c} \end{pmatrix} \\
&= \begin{pmatrix} \mathbf{1_n}'\mathbf{1_n} & \mathbf{1_n'X_c} \\ \mathbf{X_c'1_n} & \mathbf{X_c'X_c} \end{pmatrix} \\
&= \begin{pmatrix} \mathbf{1_n}'\mathbf{1_n} & 0 \\ 0 & \mathbf{X_c'X_c} \end{pmatrix} \\
\mathbf{X'X}^{-1} &= \begin{pmatrix} \mathbf{1_n}'\mathbf{1_n} & \mathbf{1_n'X_c} \\ \mathbf{X_c'1_n} & \mathbf{X_c'X_c} \end{pmatrix}^{-1} \\
&= \begin{pmatrix} n & 0 \\ 0 & (\mathbf{X_c'X_c})^{-1} \end{pmatrix} \\
(X'X)^{-1}X'Y &= \begin{pmatrix} \frac{1}{n} & 0 \\ 0 & (\mathbf{X_c'X_c})^{-1} \end{pmatrix}_{(p+1)\times(p+1)} \begin{pmatrix} \mathbf{1_n}' \\ \mathbf{X_c}' \end{pmatrix}_{(p+1)\times n} \mathbf{Y}_{n\times 1} \\
&= \begin{pmatrix} \frac{1}{n} & 0 \\ 0 & (\mathbf{X_c'X_c})^{-1} \end{pmatrix} \begin{pmatrix} \mathbf{1_n}'Y \\ \mathbf{X_c}'Y \end{pmatrix} \\
&= \begin{pmatrix} \frac{1}{n}\mathbf{1_n}'Y_{1\times 1} \\ (\mathbf{X_c'X_c})^{\mathbf{-1}}\mathbf{X_c}'Y_{p-1\times 1} \end{pmatrix} \\
&= \begin{pmatrix} \bar{Y} \\ (\mathbf{X_c'X_c})^{\mathbf{-1}}\mathbf{X_c}'Y \end{pmatrix}_{p\times 1} \\
\begin{pmatrix} \alpha \\ \beta_1 \\ \beta_2 \\ \vdots \\ \beta_{p-1} \end{pmatrix} &= \begin{pmatrix} \bar{y} \\ (\mathbf{X_c'X_c})^{\mathbf{-1}}\mathbf{X_c}'Y_{p-1\times 1} \end{pmatrix}
\end{aligned}
$$

Problem 1.5

Yes, column space of X is identical to $\begin{pmatrix} \mathbf{1_n} & \mathbf{X_c} \end{pmatrix}$ since $X_c = (I - \frac{J}{n})X$

Problem 1.6

$$
\begin{aligned}
\mathbf{P} &= \mathbf{X(X'X)^{-1}X'Y} \\
&= \begin{pmatrix} \mathbf{1_n} & X_c \end{pmatrix} \begin{pmatrix} \frac{1}{n} & 0 \\ 0 & (\mathbf{X_c'X_c})^{-1} \end{pmatrix} \begin{pmatrix} \mathbf{1_n'} \\ \mathbf{X_c'} \end{pmatrix} \\
\implies \mathbf{P} &= \mathbf{X_c(X_c'X_c)^{-1}X_c'} + \frac{\mathbf{1_n1_n'}}{n}
\end{aligned}
$$

$$
\begin{aligned}
SSE &= \epsilon'\epsilon \\
&= (Y - X\hat{\beta})'(Y - X\hat{\beta}) \\
&= Y'Y - \hat{\beta}'X'Y - Y'X\hat{\beta} + \hat{\beta}'X'X\hat{\beta} \\
&= Y'Y - 2Y'X\hat{\beta} + \hat{\beta}'X'X\{(X'X)^{-1}X'Y\} \\
&= Y'Y - 2Y'X\hat{\beta} + \hat{\beta}'IX'Y \\
&= Y'Y - 2Y'X\hat{\beta} + \hat{\beta}'X'Y \\
&= Y'Y - 2Y'X\hat{\beta} + Y'X\hat{\beta}' \\
&= Y'Y - Y'X\hat{\beta} \\
&= Y'Y - Y'\begin{pmatrix} 1_n & X_c \end{pmatrix}\begin{pmatrix} \bar{y} \\ (\mathbf{X_c'X_c})^{-\mathbf{1}}\mathbf{X_c}'Y_{p-1\times 1} \end{pmatrix} \\
&= Y'Y - Y'\begin{pmatrix} 1_n & X_c \end{pmatrix}\begin{pmatrix} \frac{1}{n}1_n'\mathbf{Y} \\ (\mathbf{X_c'X_c})^{-\mathbf{1}}\mathbf{X_c}'Y_{p-1\times 1} \end{pmatrix} \\
&= Y'(I - \frac{1_n 1_n'}{n})Y - \mathbf{Y'X_c(X_c'X_c)^{-1}X_c{}'Y} \\
&= \sum_{i=1}^{n}(Y_i - \bar{Y})^2 - \mathbf{Y'X_c(X_c'X_c)^{-1}X_c{}'Y} \\
&= \sum_{i=1}^{n}(Y_i - \bar{Y})^2 - \mathbf{Y'P_cY}
\end{aligned}
$$

Problem 1.7

$$
\begin{aligned}
Y_i^* &= Y_i - \bar{Y} \\
\mathbf{Y}^* &= \mathbf{Y} - \frac{1}{n}\mathbf{1_n 1_n' Y} \\
\sum_{i=1}^{n}(Y_i - \bar{Y})^2 &= \mathbf{Y'^* Y^*} \\
\implies SSE &= \mathbf{Y'^* Y^*} - \mathbf{Y'^* P_c Y^*}
\end{aligned}
$$