MATH 542 Homework 1

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Problem 1

$$\mathbf{1}_n = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}$$

Problem 1.a

Consider $\mathbf{a}' = \begin{pmatrix} a_1 & a_2 & \dots a_n \end{pmatrix}$

Thus,

$$\mathbf{a}'\mathbf{1}_n = \begin{pmatrix} a_1 & a_2 & \dots a_n \end{pmatrix} \times \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} = a_1 + a_2 + a_3 + \dots a_n$$

$$1_n' a = \begin{pmatrix} 1 & 1 & \dots & 1 \end{pmatrix} \times \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} = a_1 + a_2 + a_3 + \dots + a_n = \mathbf{a'1_n}$$

Problem 1.b

$$A_n I = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix} \times \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} = \begin{pmatrix} \sum_{j=1}^n a_{ij} \\ \sum_{j=1}^n a_{2j} \\ \vdots \\ \sum_{j=1}^n a_{nj} \end{pmatrix}$$

$$= \text{Column vector with row sums of } A$$

Problem 1.c

Row sum of j^{th} column of $A = \sum_{i=1}^n a_{ij}$

Column sum of i^{th} row of $A = \sum_{j=1}^n a_{ij}$

$$
\begin{aligned}
1_n' A &= \begin{pmatrix} 1 & 1 & \dots 1 \end{pmatrix} \times \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix} \\
&= \begin{pmatrix} \sum_{i=1}^n a_{i1} & \sum_{i=1}^n a_{i2} & \sum_{i=1}^n a_{i3} & \cdots \sum_{i=1}^n a_{in} \end{pmatrix} \\
&= \text{Row vector with elements as column sums of A}
\end{aligned}
$$

Problem 2

$$
\begin{aligned}
a_1 &= \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} \\
A_2 &= \begin{pmatrix} 1 & 2 \\ 2 & 0 \\ 0 & 1 \end{pmatrix} \\
b_1' &= \begin{pmatrix} 1 & 1 & 1 & 0 \end{pmatrix} \\
B_2 &= \begin{pmatrix} 2 & 1 & 1 & 2 \\ 2 & 3 & 1 & 2 \end{pmatrix}
\end{aligned}
$$

$$
AB = a_1 b_1' + A_2 B_2
$$

$$
a_1 b_1' = \begin{pmatrix} 2 & 2 & 2 & 0 \\ 3 & 3 & 3 & 0 \\ 1 & 1 & 1 & 0 \end{pmatrix}
$$

Similarly,

$$
A_2 B_2 = \begin{pmatrix} 6 & 7 & 3 & 6 \\ 4 & 2 & 2 & 4 \\ 2 & 3 & 1 & 2 \end{pmatrix}
$$

And hence $AB = a_1 b_1' + A_2 B_2 = \begin{pmatrix} 6 & 7 & 3 & 6 \\ 4 & 2 & 2 & 4 \\ 2 & 3 & 1 & 2 \end{pmatrix}$

Problem 3

A is $n \times p$

Problem 3.a

AA' is symmetric if $AA' = A'A$

Proof: $(AA')' = (A')'A' = AA'$ and hence $AA'_{n\times n}$ is symmetric

Similarly, for $A'A$ consider the following:

$(A'A)' = A'A'' = A'A$ and hence $A'A_{p\times p}$ is symmetric too.

Problem 3.b

$(A'A)_{ij} = \sum_{k=1}^{n} a_{ki}a_{kj}$ and hence any diagonal element of $A'A$ is a sum of perfect squares(the case when $i = j \implies (A'A)_{ii} = \sum_{k=1}^{n} a_{ki}^2$)

If $\sum_{k=1}^{n} a_{ki}^2 = 0 \implies a_{ki} = 0\ \forall 1 \le k \le n\ \forall i$ Hence A is a zero matrix.

Problem 4

Problem 4.a

$$A_2 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$A_3 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$A_4 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Problem 4.b

Rank of V_2:

$$V_2 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\xrightarrow{\text{C2-C1}} \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 1 & -1 \\ 0 & 0 \end{pmatrix}$$

which cannot be reduced further, hence rank $= min(3, 2) = 2$

Rank of V_3:

$$A_3 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\xrightarrow{\text{R1-R2}} \begin{pmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Thus rank of $V_3 = 3$

Problem 4.c

Basis for V_2(maximising zeroes by doing a $C1 - C2$ operation): $\left(\begin{pmatrix} 0 & 1 & 0 \end{pmatrix}', \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}' \right)$

Basis for V_3 (maximised zeroes): $\left(\begin{pmatrix} 1 & 0 & 0 \end{pmatrix}', \begin{pmatrix} 1 & 1 & 0 \end{pmatrix}', \begin{pmatrix} 0 & 0 & 1 \end{pmatrix}' \right)$