MATH 542 Homework 2

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Problem 1

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$$

Consider $C = A^{-1}$ so that $CA = I$

$$C = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix}$$

$$\begin{aligned} CA &= \begin{pmatrix} A_{11}C_{11} + A_{12}C_{21} & A_{11}C_{12} + A_{12}C_{22} \\ A_{21}C_{11} + A_{22}C_{21} & A_{21}C_{12} + A_{22}C_{22} \end{pmatrix} \\ &= \begin{pmatrix} I_k & O_{n-k} \\ O_{n-k} & I_k \end{pmatrix} \end{aligned}$$

Thus,

$$\begin{aligned} A_{11}C_{11} + A_{12}C_{21} &= I_k \\ A_{11}C_{12} + A_{12}C_{22} &= O_{n-k} \\ A_{21}C_{11} + A_{22}C_{21} &= O_{n-k} \\ A_{21}C_{12} + A_{22}C_{22} &= I_k \end{aligned}$$

Thus,

$$\begin{aligned} C_{11} &= -A_{11}^{-1}(I - A_{12}C_{21}) \\ C_{12} &= -A_{11}^{-1}A_{12}C_{22} \\ C_{21} &= -A_{22}^{-1}A_{21}C_{11} \\ C_{22} &= A_{22}^{-1}(I - A_{21}C_{12}) \end{aligned}$$

Substituting for C_{12} and C_{21} :

$$\begin{aligned} A_{11}C_{11} - A_{12}A_{22}^{-1}A_{21}C_{11} &= I_k \\ \implies C_{11} &= (A_{11} - A_{12}A_{22}^{-1}A_{21})^{-1} \end{aligned}$$

Similarly,

$$\begin{aligned} -A_{21}A_{11}^{-1}A_{12}C_{22} + A_{22}C_{22} &= I_k \\ C_{22} &= (A_{22} - A_{21}A_{11}^{-1}A_{12})^{-1} \\ &= B \end{aligned}$$

Thus,

$$\begin{aligned} B &= (A_{22} - A_{21}A_{11}^{-1}A_{12}) \\ C_{11} &= (A_{11} - A_{12}A_{22}^{-1}A_{21})^{-1} \\ C_{21} &= -A_{22}^{-1}A_{21}(A_{11} - A_{12}A_{22}^{-1}A_{21})^{-1} \\ C_{22} &= (A_{22} - A_{21}A_{11}^{-1}A_{12})^{-1} = B^{-1} \\ C_{12} &= -A_{11}^{-1}A_{12}C_{22} \end{aligned}$$

Now we use Woodbury matrix identity $(A+UCV)^{-1} = A^{-1} - A^{-1}U(C^{-1} + VA^{-1}U)^{-1}VA^{-1}$

Using which,

$$\begin{aligned} C_{11} &= (A_{11} - A_{12}A_{22}^{-1}A_{21})^{-1} \\ &= A_{11}^{-1} - A_{11}^{-1}(-A_{12})(A_{22}^{-1} + -A_{21}A_{11}^{-1}A_{12})^{-1}A_{21}A_{11}^{-1} \\ &= A_{11} + A_{11}^{-1}A_{12}B^{-1}A_{21}A_{11}^{-1} \end{aligned}$$

$$\begin{aligned} C_{12} &= -A_{11}^{-1}A_{12}C_{22} \\ &= -A_{12}^{-1}A_{12}B^{-1} \end{aligned}$$

Problem 1.b

For this part we substitute $k = 1$

$$\begin{aligned} B_{1\times 1} &= a_{22} - a'_{12}A_{11}^{-1}a_{12} \\ C_{11} &= A_{11} + A_{11}a_{12}B^{-1}a'_{21}A_{11}^{-1} \\ C_{12} &= -A_{11}^{-1}a_{12}B^{-1} \\ C_{21} &= -B^{-1}a'_{12}A_{11}^{-1} \\ C_{22} &= B^{-1} \end{aligned}$$

Problem 2

Problem 2.a

$$X'X = \begin{pmatrix} 2J & J & J \\ J & J & 0 \\ J & 0 & J \end{pmatrix} = J\begin{pmatrix} 2 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

Problem 2.b

$X'X$ is not invertible(Perform $C1 - C3$)

$$(X'X) \times (X'X)_1^- = J \begin{pmatrix} 2 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \times XX_1^- = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & -1 & 0 \end{pmatrix}$$

Also, $(X'X) \times (X'X)_2^- = J \begin{pmatrix} 2 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \times XX_2^- = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Problem 2.c

$P = X(X'X)_1^- X' = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} = X(X'X)_2^- X' = X'X$ and thus P is unique

Problem 3

$$X = \begin{pmatrix} 1 & -1 & 1 \\ -1 & 0 & 2 \\ 1 & 1 & 1 \end{pmatrix}$$

Consider $X^T = \begin{pmatrix} 1 & -1 & 1 \\ -1 & 0 & 1 \\ 1 & 2 & 1 \end{pmatrix}$

Finding the inverse gives $X^{-1} = \begin{pmatrix} 1 & -1 & 1 \\ -1 & 0 & 2 \\ 1 & 1 & 1 \end{pmatrix}$ In fact $X^T X = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 6 \end{pmatrix}$ and $XX^T = \begin{pmatrix} 3 & 1 & 1 \\ 1 & 5 & 1 \\ 1 & 1 & 3 \end{pmatrix}$

Clearly $X^T \neq X^{-1}$ and this leads us to conclude that X is **not** an orthonormal matrix.

$$C = \begin{pmatrix} 1/\sqrt{3} & -1/\sqrt{2} & 1/\sqrt{6} \\ -1/\sqrt{3} & 0/\sqrt{2} & 2/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \end{pmatrix}$$

$$C^T = \begin{pmatrix} 1/\sqrt{3} & -1/\sqrt{3} & 1/\sqrt{3} \\ -1/\sqrt{2} & 0/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{6} & 2/\sqrt{6} & 1/\sqrt{6} \end{pmatrix}$$

Thus,

$$CC^T = C^T C = I_3 \text{ (Skipped calculations, did in R)}$$

Problem 4

Problem 4.a

Using the property $\det(XY) = \det(X) \times \det(Y)$ we have $\det(A^2) = \det(A) \times \det(A) = (\det(A))^2$

0.1 Problem 4.b

$A = P\Lambda P^T$ such that Λ is a diagonal matrix with its diagonals as the eigen values and P is an orthonormal matrix thus, $PP^T = I$ and $\det(P) = +1\ or 1$

$\det(A) = \det(P\Lambda P^T) = \det(P)\det(\Lambda)\det(P^T) = \det(\Lambda)\det(P)^2 = \det(\Lambda) = \prod \lambda_i$ Using $tr(AB) = tr(BA)$ $tr(A) = tr(P\Lambda P^T) = tr(\lambda P^T P) = tr(\Lambda \times I) = tr(\Lambda) = \sum \lambda_i$