MATH 542 Homework 3

Saket Choudhary
skchoudh@usc.edu

February 4, 2016

Problem 1

Given: $A_{n\times n}$ is idempotent $\implies$ $AA = A$; P is non-singular and $C_{n\times n}$ is orthogonal $\implies CC^T = C^TC = I$

1.a

$$\begin{aligned}(I-A)(I-A) &= I - A - A + AA\\ &= I - 2A + A \;\text{ using } AA = A\\ &= I - A\end{aligned}$$

Hence $I - A$ is idempotent

1.b

$$\begin{aligned}A(I-A) &= A - AA\\ &= A - A \;\text{ using } AA = A\\ &= 0\end{aligned}$$

Similarly,

$$\begin{aligned}(I-A)A &= A - AA\\ &= A - A \;\text{ using } AA = A\\ &= 0\end{aligned}$$

1.c

$$\begin{aligned}(P^{-1}AP)(P^{-1}AP) &= P^{-1}APP^{-1}AP\\ &= P^{-1}AAP \text{ since } PP^{-1} = I\\ &= P^{-1}AP \;\text{ using } AA = A\end{aligned}$$

Hence $P^{-1}AP$ is idempotent

1.d

$$\begin{aligned}(C'AC)(C'AC) &= C'ACC'AC \\ &= C'AAC \text{ since } CC' = C'C = I \\ &= C'AC \quad \text{using } AA = A\end{aligned}$$

1.e

A is a projection matrix and $C'C = C'C = I$ For $C'AC$ to be a projection matrix $C'ACz = z\ \forall z \in S$ for some vector space S

For some $z \in S$:

$$\begin{aligned}C'ACz &= C'Ay \text{ where } y = Az \\ &= C'Ay \\ &= C'y \text{ since } A \text{ is projection matrix} \\ &= C'Cz \\ &= z \quad \text{since } C'C = I\end{aligned}$$

Thus, $C'ACz \in S$ and $C'ACz = z$ and hence $C'AC$ is a projection matrix

Problem 2

2.a

$$\begin{aligned}Q(x_1, x_2, x_3) &= 12x_1^2 + 3x_2^2 + 3x_3^2 + 2x_1x_2 - 10x_1x_3 + 4x_2x_3 \\ &= 3(2x_1 + x_2 + x_3)^2 - (12x_1x_2 + 6x_2x_3 + 12x_1x_3) + 2x_1x_2 - 10x_1x_3 + 4x_2x_3 \\ &= 3(2x_1 + x_2 + x_3)^2 - 10x_1x_2 - 2x_2x_3 - 22x_1x_3 \\ &= (x_1 + x_2)^2 + 2(x_2 + x_3)^2 + (5x_1 - x_3)^2 - 14x_1^2 \\ &= (x_1 + x_2)^2 + 2(x_2 + x_3)^2 + x_3^2 + 11x_1^2 - 10x_1x_3 \\ &= (x_1 + x_2)^2 + 2(x_2 + x_3)^2 + 25(x_1 - x_3/5)^2 - 14x_1^2\end{aligned}$$

and hence $Q(x_1, x_2, x_3)$ is positive defnite, (the coefficient of the negative term(-x_1^2) is less than the positive coefficient of x_1^2

2.b

$$x'Ax = \begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix} \begin{pmatrix} 2 & 1 & 2 \\ 1 & 2 & 1 \\ 2 & 1 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$\begin{aligned}&= (2x_1 + x_2 + x_3)x_1 + (x_1 + 2x_2 + x_3)x_2 + (2x_1 + x_2 + 4x_3)x_3 \\ &= 2x_1^2 + 2x_2^2 + 4x_3^2 + 2x_1x_2 + 3x_1x_3 + 2x_2x_3 \\ &= (x_1 + x_2)^2 + (x_2 + x_3)^2 + (x_1 + \frac{3}{2}x_2)^2 + \frac{3}{4}x_2^2 \\ &> 0\end{aligned}$$

A is positive definite

2.c

$$
\begin{aligned}
x'Ax = \begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 1 \\ 3 & 1 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \\
&= (x_1 + 2x_2 + 3x_3)x_1 + (2x_1 + x_2 + x_3)x_2 + (3x_1 + x_2 - 2x_3)x_3 \\
&= x_1^2 + x_2^2 - 2x_3^2 + 4x_1x_2 + 6x_1x_3 + 2x_2x_3 \\
&= (x_2 + x_3)^2 + 3(x_1 + x_3)^2 + 2(x_1 + x_2)^2 - 6x_1^2 - 6x_3^2 - 4x_1^2 - 2x
\end{aligned}
$$

A is neither positive definite nor positive semidefinite.

Problem 3

$$
\begin{pmatrix} a & b \\ c & d \end{pmatrix} \times \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}
$$

$$
\begin{pmatrix} a^2 + b^2 & ac + bd \\ ac + bd & c^2 + d^2 \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}
$$

$$
\begin{aligned}
a^2 + b^2 &= 2 \\
ac + bd &= -1 \\
c^2 + d^2 &= 2
\end{aligned}
$$

Let $a = 1, b = 1 \implies c + d = 1$

$$
\begin{aligned}
c^2 + d^2 &= 2 \\
c^2 + (-1 - c)^2 &= 2 \\
2c^2 + 2c - 1 &= 0 \\
c &= \frac{-2 \pm \sqrt{4 + 8}}{4} \\
c &= \frac{-1 \pm \sqrt{3}}{2} \\
d &= \frac{-1 \pm \sqrt{3}}{2}
\end{aligned}
$$

and hence one possible B is:

$$
B = \begin{pmatrix} 1 & 1 \\ \frac{-1-\sqrt{3}}{2} & \frac{-1+\sqrt{3}}{2} \end{pmatrix}
$$

Problem 4

$G = (X'X)^-$

Let $Y = (X'X)$ to prove GYG is symmetric we need to show $(GYG)^T = GYG$

NOTE: $Y^T = (X^TX)^T = X^TX = Y$

Using $YGY = Y$, we get:

$$\begin{aligned} YGY &= Y \\ YGYG &= YG \\ \implies GYG &= G \end{aligned}$$

Now, again consider YGY and take transpose on both sides:

$$\begin{aligned} (YGY)^T &= Y^T \\ Y^TG^TY^T &= Y^T \\ YG^TY &= Y \text{ using } Y^T = Y \\ \implies G &= G^T \text{ since } YG^TY = YGY = Y \text{Also see NOTE below} \end{aligned}$$

Now consider GYG

$$\begin{aligned} (GYG)^T &= G^TY^TG^T \\ &= GYG \text{ using } G = G^T and Y = Y^T \end{aligned}$$

NOTE:

1.

However, this case is only true for the case when $G = G^T$ i.e. the generalized inverse in unique. I could not prove it for the general case

To show GYG is a generalized inverse of Y we need to show $YGYGY = Y$

$$\begin{aligned} Y(GYG)Y &= YG(YGY) \\ &= YGY \text{ using } YGY = Y, \text{ G is generalised inverse of Y} \\ &= Y \text{ using definition of generalised inverse for } G \end{aligned}$$

$\implies$ GYG is generalised inverse of $Y = X^TX$

Problem 5

Given: $E[X] = 1$ and $Var(X) = E[X^2] - E[X]^2 = 5$ Using $E[X] = 1$ we get, $E[X^2] = 5 + E[X]^2 = 6$

5.a

$$
\begin{aligned}
E[(1+2X)^2] &= E[1+4X+4X^2] \\
&= E[1] + E[4X] + E[4X^2] \\
&= 1 + 4E[X] + 4E[X^2] \\
&= 1 + 4 + 4(6) \\
&= 30
\end{aligned}
$$

5.b

$$
\begin{aligned}
var(3+4X) &= Var(3) + Var(4X) + 2Cov(3, 4X) \\
&= 0 + 16Var(X) + 2(0) \\
&= 80
\end{aligned}
$$