MATH 542 Homework 4

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1a. Problem 1

$$
\begin{aligned}
E[(X-a)(X-a)'] &= E[(X-a)(X'-a')] \\
&= E[XX' - Xa' - aX' + aa'] \\
&= E[XX'] - E[Xa'] - E[aX'] + E[aa'] \\
&= E[XX'] - E[X]a' - aE[X'] + E[aa'] \\
&= (Var[X] + E[X]E[X]') - E[X]a' - aE[X'] + aa' \\
&= Var[X] + E[X]E[X]' - E[X]a' - aE[X]' + aa' \quad \text{since } E[X'] = E[X]' \\
&= Var[X] + (E[X]-a)(E[X]'-a') \\
&= Var[X] + (E[X]-a)(E[X]-a)'
\end{aligned}
$$

$$Var[X] = \sum = (\sigma_{ij})$$

$$||X-a||^2_{1\times 1} = (X-a)'_{1\times r}(X-a)_{1\times r}$$

And hence(replace X with X' and a with a'):

$$
\begin{aligned}
E[||X-a||^2] &= E[(X-a)'(X-a)] \\
&= E[X'X] - E[X']a - a'E[X] + a'a \\
&= \sum_i E[X_i^2] - E[X']a - a'E[X] + a'a \\
&= \sum_i (Var[X_i] + E[X_i]^2) - E[X']a - a'E[X] + a'a \\
&= \sum_i Var[X_i] + E[X']E[X] - E[X']a - a'E[X] + a'a \text{ since } \sum_i E[X_i]^2 = E[X'X] \\
&= \sum_i Var[X_i] + E[X]'E[X] - E[X]'a - a'E[X] + a'a \text{ since } \sum_i E[X_i]^2 = E[X'X] \\
&= \sum_i Var[X_i] + (E[X]-a)'(E[X]-a) \\
&= \sum_i \sigma_i + ||E[X]-a||^2
\end{aligned}
$$

1a. Problem 2

Fact: $X - a - E[X-a] = X - E[X]$

$$\begin{aligned} Cov[X-a, Y-b] &= E[(X-a-E[X-a])(Y-b-E[Y-b])')] \\ &= E[(X-E[X])(Y-E[Y])'] \\ &= Cov[X,Y] \end{aligned}$$

1a. Problem 3

$Y_i = X_i - X_{i-1}$

$Cov[Y_i, Y_j] = 0$ for $i \neq j$

Consider the vector $(Y_1, Y_2, Y_3, \ldots, Y_n)' = (X_1, X_2 - X_1, X_3 - X_2, \ldots, X_n - X_{n-1})'$

We make use of $Var(AX) = AVar(X)A'$.

To find A, consider the vectors $(Y_1, Y_2, Y_3, \ldots, Y_n)' = (X_1, X_2 - X_1, X_3 - X_2, \ldots, X_n - X_{n-1})'$

$$\begin{aligned} \begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \\ \vdots \\ Y_n \end{pmatrix} &= \begin{pmatrix} X_1 \\ X_2 - X_1 \\ X_3 - X_2 \\ \vdots \\ X_n - X_{n-1} \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 & \ldots & 0 & 0 \\ -1 & 1 & 0 & \ldots & 0 & 0 \\ 0 & -1 & 1 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & -1 & 1 \end{pmatrix} \times \begin{pmatrix} X_1 \\ X_2 \\ X_3 \\ \vdots \\ X_n \end{pmatrix} \end{aligned}$$

Hence $A = \begin{pmatrix} 1 & 0 & 0 & \ldots & 0 & 0 \\ -1 & 1 & 0 & \ldots & 0 & 0 \\ 0 & -1 & 1 & \ldots & 0 & 0 \\ \vdots & & & & & \\ 0 & 0 & 0 & \ldots & -1 & 1 \end{pmatrix}$

Now using $Var(Y) = AVar(X)A'$ we get

$$Var(X) = A^{-1}Var(Y)A'^{-1}$$

$Var(Y) = I_{n \times n}$ and hence $Var(X) = A^{-1}A'^{-1} = BB^T$ where $B = A^{-1}$

Problem 4

$X_{i+1} = \rho X_i$

Consider:

$$\begin{pmatrix} X_1 \\ X_2 \\ X_3 \\ \vdots \\ X_n \end{pmatrix} = \begin{pmatrix} X_1 \\ \rho X_1 \\ \rho X_2 \\ \vdots \\ \rho X_{n-1} \end{pmatrix} = \begin{pmatrix} 1 \\ \rho \\ \rho^2 \\ \vdots \\ \rho^{n-1} \end{pmatrix} X_1$$

Let $A = \begin{pmatrix} 1 & \rho & \rho^2 & \dots & \rho^{n-1} \end{pmatrix}'$ and hence variance $Var[X] = AVar(X_1)A' = \sigma^2 AA'$

$$Var[X] = \begin{pmatrix} 1 & \rho & \rho^2 & \dots & \rho^{n-1} \\ \rho & \rho^2 & \rho^3 & \dots & \rho^n \\ \rho^2 & \rho^3 & \rho^4 & \dots & \rho^{n+1} \\ \vdots & \vdots & \vdots & & \vdots \\ \rho^n & \rho^{n+1} & \rho^{n+2} & \dots & \rho^{2n-2} \end{pmatrix}$$

1b. Problem 1

$$X_1^2 + 2X_1X_2 - 4X_2X_3 + X_3^2 = (X_1 + X_2)X_1 + (X_1 - 2X_3)X_2 + (-2X_2 + X_3)X_3$$
$$= \begin{pmatrix} X_1 & X_2 & X_2 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & -2 \\ 0 & -2 & 1 \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix}$$

$$X = \begin{pmatrix} X_1 & X_2 & X_3 \end{pmatrix}'$$

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & -2 \\ 0 & -2 & 1 \end{pmatrix}$$

$$A \sum = AVar[X] = \sigma^2 \begin{pmatrix} 1 & 1 & \frac{1}{4} \\ 1 & -\frac{1}{2} & -2 \\ 0 & -\frac{7}{4} & \frac{1}{2} \end{pmatrix}$$

Thus,

$$E[X'AX] = tr(A \sum) + \mu' A \mu = \sigma^2 + \mu' A \mu$$

1b. Problem 2

$$\begin{aligned}\sum_i (X_i - \bar{X})^2 &= \sum_i (X_i^2 - 2X_i\bar{X} + \bar{X}^2) \\ &= \sum_i X_i^2 - 2\sum_i X_i\bar{X} + \bar{X}^2 \\ &= \sum_i X_i^2 - n\bar{X}^2\end{aligned}$$

Now $\sum_i X_i^2 = X'X$ and $\bar{X} = \frac{1}{n}\sum_i X_i = \frac{1}{n}\mathbf{1}'X = \frac{1}{n}X'\mathbf{1}$
Hence,

$$\begin{aligned}\sum_i (X_i - \bar{X})^2 &= \sum_i X_i^2 - n\bar{X}^2 \\ &= X'X - n\frac{1}{n^2}(X'\mathbf{1}\mathbf{1}'X) \\ &= X'X - \frac{1}{n}(X'\mathbf{1}\mathbf{1}'X) \\ &= X'(\mathbf{I} - \frac{1}{n}\mathbf{1}\mathbf{1}')X\end{aligned}$$

Let $A = (\mathbf{I} - \frac{1}{n}\mathbf{1}\mathbf{1}')$
Now using $E[X'AX] = tr(A\sum) + \mu' A\mu$ we have:

$$\begin{aligned}A\sum &= \begin{pmatrix} 1-1/n & -1/n & -1/n & \dots & -1/n \\ -1/n & 1-1/n & -1/n & \dots & -1/n \\ \vdots & & & & \\ -1/n & -1/n & -1/n & \dots & 1-1/n \end{pmatrix} \times diag(\sigma_1^2, \sigma_2^2, \dots, \sigma_n^2) \\ &= (1-\frac{1}{n})\sum_i \sigma_i^2\end{aligned}$$

Also,

$$\mu' A = \begin{pmatrix} \mu & \mu & \dots & \mu \end{pmatrix} \times \begin{pmatrix} 1-1/n & -1/n & -1/n & \dots & -1/n \\ -1/n & 1-1/n & -1/n & \dots & -1/n \\ \vdots & & & & \\ -1/n & -1/n & -1/n & \dots & 1-1/n \end{pmatrix} = 0$$

and hence $\mu' A\mu = 0$

$$E[\sum_i (X_i - \bar{X})^2] = (1-\frac{1}{n})\sum_i \sigma_i^2$$
$$E[\frac{1}{n(n-1)}\sum_i (X_i - \bar{X})^2] = \frac{1}{n^2}\sum_i \sigma_i^2$$

Finally,

$$var(\bar{X}) = Var(\frac{1}{n}\sum_i X_i)$$

$$= \frac{1}{n^2}\sum_i Var(X_i)$$

since X_i are mutually independent

$$= \frac{1}{n^2}\sum_i \sigma_i^2$$

$$= E[\frac{1}{n(n-1)}\sum_i (X_i - \bar{X})^2]$$

1b. Problem 3

Given: $\bar{X_w} = \sum_i w_i X_i$ and $\sum w_i = 1$

$$\begin{aligned} Var(\bar{X_w}) &= Var(\sum w_i X_i) \\ &= \sum w_i^2 Var(X_i) \quad \text{since } X_i \text{ are mutually independent} \\ &= \sum w_i^2 \sigma_i^2 \end{aligned}$$

Now consider,

$$\text{minimize} \sum w_i^2 \sigma_i^2 \text{ subject to } \sum_i w_i = 1$$

We consider the following lagrange formulation:

$$min_w f(\mathbf{w}) = \sum_i w_i^2 \sigma_i^2 + \lambda(\sum_i w_i - 1)$$

Now to find optimal λ, we solve $\frac{\partial f(\mathbf{w})}{\partial w_i} = 0$

$$\frac{\partial f()}{\partial w_i} = 2 w_i \sigma_i^2 + \lambda = 0$$

$$\implies w_i = -\frac{\lambda}{2\sigma_i^2}$$

Thus, $w_i = -\frac{\lambda}{2\sigma_i^2}$ or $w_i \propto \frac{1}{\sigma_i^2}$

Using $\sum_i w_i = 1$ we get:

$$\sum_i w_i = 1$$
$$\sum_i \frac{\lambda}{-2\sigma_i^2} = 1$$
$$\implies \lambda = \frac{-2}{\sum_i 1/\sigma_i^2}$$
$$\implies w_i = \frac{1}{\sigma_i^2 \sum_i (1/\sigma_i^2)}$$
$$\implies f_{min}(w) = \sum_i (\frac{1}{\sigma_i^2 \sum_i (1/\sigma_i^2)})\sigma_i^2$$
$$v_{min} = \frac{1}{\sum_i (1/\sigma_i^2)}$$

Part b

$$\begin{aligned}\sum_i w_i(X_i - \bar{X_w})^2 &= \sum_i w_i(X_i^2 - 2X_i\bar{X_w} + \bar{X_w}^2)\\ &= \sum_i w_i X_i^2 - 2\bar{X_w}\sum_i w_i X_i + \bar{X}_w^2\\ &= \sum_i w_i X_i^2 - 2\bar{X_w}^2 + \bar{X_w}^2\\ &= \sum_i w_i X_i^2 - \bar{X_w}^2\end{aligned}$$

Now, we rewrite $\sum_i w_i X_i^2 = X'\Lambda X$ where $\Lambda = diag(w_1, w_2, \dots, w_n)$ and $\bar{X_w} = \sum_i w_i X_i = X'w = w'X$

$$\begin{aligned}\bar{X_w}^2 &= (\sum_i w_i X_i)^2\\ &= X'ww'X\end{aligned}$$

and hence $\sum_i w_i(X_i - \bar{X_w})^2 = X'(\Lambda - ww')X$

Define $A = \Lambda - ww' = \begin{pmatrix} w_1 - w_1^2 & w_1w_2 & w_1w_3 & \dots & w_1w_n \\ w_2w_1 & w_2 - w_2^2 & w_2w_3 & \dots & w_2w_n \\ \vdots & \vdots & \vdots & \vdots & \\ w_nw_1 & w_nw_2 & w_nw_3 & \dots w_n - w_n^2 & \end{pmatrix}$

so that

$$E[\sum_i w_i(X_i - \bar{X_w})^2] = E[X'AX] = tr(A\sum) + \mu' A\mu$$

$$tr(A\sum) = \sum_i (w_i - w_i^2)\sigma_i^2 = \sum_i w_i\sigma_i^2 - w_i(w_i\sigma_i^2) = \sum_i (a - aw_i) = na - a$$

and the $(1,1)$ element of the matrix $\mu' A$ is given by:

$$\begin{aligned}
\mu' A_1 &= \mu(w_1 - w_1^2 - w_1 \sum_{i=2} w_i) \\
&= w_1 - w_1^2 - w_1(1 - w_1) \\
&= 0
\end{aligned}$$

and hence it's essentially a zero matrix(other elements are zero similarly)
Also,

$$v_{min} = \frac{1}{\sum_i 1/\sigma_i^2} = \frac{1}{\sum_i \frac{w_i}{a}} = a$$

Thus,

$$\begin{aligned}
E[S_w^2] &= \frac{1}{n-1} tr(A \sum) + \mu' A \mu \\
&= \frac{1}{n-1}(na - a) + 0 \\
&= a \\
&= v_{min}
\end{aligned}$$