MATH 542 Homework 5

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Problem 1

Problem 1a.

Using variance-covariance expansion:

$$\begin{aligned}
Var(X_1 - 2X_2 + X_3) &= Var(X_1) + Var(-2X_2) + Var(X_3) + 2Cov(X_1, -2X_2) + 2Cov(-2X_2, X_3) + 2Cov(X_3, \\
&= Var(X_1) + 4Var(X_2) + Var(X_3) - 4Cov(X_1, X_2) - 4Cov(X_2, X_3) + 2Cov(X_3, X_1) \\
&= 5 + 4(3) + 3 - 4(2) - 4(0) + 2(3) \\
&= 18
\end{aligned}$$

$$\begin{aligned}
Y &= \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} \\
&= \begin{pmatrix} X_1 + X_2 \\ X_1 + X_2 + X_3 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix}
\end{aligned}$$

Now using $Var(AX) = AVar(X)A'$

$$\begin{aligned}
Var(Y) &= \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} Var(X) \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 12 & 15 \\ 15 & 21 \end{pmatrix}
\end{aligned}$$

Ex2a Problem 1

$f(y_1, y_2) = k^{-1} \exp(-\frac{1}{2}(2y_1^2 + y_2^2 + 2y_1y_2 - 22y_1 - 14y_2 + 65))$

$$
\begin{aligned}
2y_1^2 + y_2^2 + 2y_1y_2 - 22y_1 - 14y_2 + 65 &= \begin{pmatrix} y_1 - \mu_1 & y_2 - \mu_2 \end{pmatrix} \begin{pmatrix} a & b \\ b & c \end{pmatrix} \begin{pmatrix} y_1 - \mu_1 \\ y_2 - \mu_2 \end{pmatrix} \\
&= a(y_1 - \mu_1)^2 + 2b(y_1 - \mu_1)(y_2 - \mu_2) + c(y_2 - \mu_2)^2 \\
&= ay_1^2 + cy_2^2 + 2by_1y_2 - y_1(2a\mu_1 + 2b\mu_2) - y_2(2b\mu_1 + 2c\mu_2) + (a\mu_1^2 + 2b\mu_1\mu_2
\end{aligned}
$$

Now comparing the coefficient of $y_1^2 \implies a = 2$
Comparing coefficient of $y_2^2 \implies c = 1$
Comparing coefficient of $y_1y_2 \implies b = 1$
Comparing coefficient of $y_1 \implies 4\mu_1 + 2\mu_2 = 22$
Comparing coefficient of $y_2 \implies 2\mu_1 + 2\mu_2 = 14$
Thus, $\mu_1 = 4$ and $\mu_2 = 3$
Check: $4\mu_1^2 + 2\mu_1\mu_2 + \mu_2^2 = 2(16) + 24 + 9 = 65$
and hence $\Sigma^{-1} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$
$det(\Sigma^{-1}) = 1$
$\Sigma = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}$
Thus, $k^{-1} = \frac{1}{\sqrt{2\pi det(\Sigma)}}^{2/2} = \frac{1}{2\pi}$ Thus, $k = 2\pi$

2a Problem 1b

$$
\begin{aligned}
E[Y] &= \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix} \\
&= \begin{pmatrix} 4 \\ 3 \end{pmatrix}
\end{aligned}
$$

$$
\begin{aligned}
Var[Y] &= \begin{pmatrix} a & b \\ b & c \end{pmatrix} \\
&= \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}
\end{aligned}
$$

Ex2a Problem 3(b)

$$\Sigma = \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}$$

Determining eigen values:

$$
\begin{aligned}
det(\Sigma - \lambda I) &= 0 \\
(1 - \lambda)^2 &= \rho^2 \\
\lambda &= 1 \pm \rho
\end{aligned}
$$

And the corresponding eigen values:

$$\begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \lambda \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

One set of eigen vectors are given by: for $\lambda_1 = 1+\rho$: $\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}$ and for $\lambda_2 = 1-\rho$: $\frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}$

Thus using eigen decomposition Σ can be rewritten as:

$$\begin{aligned}\Sigma &= A\Lambda A' \\ &= \frac{1}{2}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\begin{pmatrix}1+\rho & 0\\ 0 & 1-\rho\end{pmatrix}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\end{aligned}$$

And hence $\Sigma^{1/2} = A\Lambda^{1/2}A' = \frac{1}{2}\begin{pmatrix}\sqrt{1+\rho}+\sqrt{1-\rho} & \sqrt{1+\rho}-\sqrt{1-\rho}\\ \sqrt{1+\rho}-\sqrt{1-\rho} & \sqrt{1+\rho}+\sqrt{1-\rho}\end{pmatrix}$

Ex2b Problem 2

$Y_i = \begin{pmatrix}0 & 0 & \dots & 1_i & 0\dots 0\end{pmatrix} Y = a_i'Y$

Since $Y_i \sim N(\mu, \Sigma)$ Using Theorem 2.2 $Y_i \sim N(a_i'\mu, a_i'\Sigma a_i) = N(\mu_i, \sigma_{ii})$

Ex2b Problem 3

Since Y_1, Y_2, Y_3 and $Y_1 - Y_2$ are both normal, their joint distribution is normal too. Consider: $A = \begin{pmatrix}1 & 1 & 1\\ 1 & -1 & 0\end{pmatrix}$

Now, $Z = \begin{pmatrix}Z_1\\ Z_2\end{pmatrix} = AY$

and hence $Z \sim N(A\mu, A\Sigma A')$ $A\mu = \begin{pmatrix}5\\1\end{pmatrix}$

$A\Sigma A' = \begin{pmatrix}10 & 0\\ 0 & 3\end{pmatrix}$

Since Z_1 and Z_2 are normal, and they are independent $\sigma_{12} = 0$ so the joint distribution is given by their product.

$\sigma_1^2 = 10$; $\sigma_2^2 = 3$ $\mu_1 = 5$, $\mu_2 = 1$

$$f(Z_1, Z_2) = \frac{1}{2\pi\sigma_1\sigma_2}\exp(-\frac{(Z_1-\mu_1)^2}{2\sigma_1^2} - \frac{(Z_2-\mu_2)^2}{2\sigma_2^2})$$

Ex2b Problem 6

Define $U_1 = Y_1 + Y_2$ and $U_2 = Y_1 - Y_2$ where $U_i \sim N(0,1)$

$Cov(U_1, U_2) = 0$

Rearranging gives:

$$\begin{aligned}Y_1 &= \frac{1}{2}(U_1 + U_2)\\ Y_2 &= \frac{1}{2}(U_1 - U_2)\end{aligned}$$

Thus, $Y_i \sim N(0, \frac{1}{4}$

Since any vector $a'Y$ has a univariate normal distribution(mean=0) using Theorem 2.3, we see that $Y \sim N(\mu, \Sigma)$ where

$$\begin{aligned} Y &= \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} \\ \mu &= \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \end{aligned}$$

To find Σ:

$$\begin{aligned} Cov(U_1, U_2) = 0 \\ Cov(Y_1 + Y_2, Y_1 - Y_2) = 0 \\ Cov(Y_1, Y_1) + Cov(Y_1, Y_2) + Cov(Y_2, Y_1) + Cov(Y_2, -Y_2) = 0 \\ \sigma_{11} + 2\sigma_{12} - \sigma_{22} = 0 \implies \sigma_{12} \quad &= 0 \text{ using } \sigma_{11} = \sigma_{22} = 1 \end{aligned}$$

Thus, Y_1, Y_2 have a bivariate normal ditribution. with $\mu = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ and $\Sigma = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

Ex2b Problem 8

$$\begin{pmatrix} \bar{Y} & Y_1 - \bar{Y} & Y_2 - \bar{Y_3} \ldots Y_n - \bar{Y} \end{pmatrix}' = \begin{pmatrix} 1/n & 1/n & 1/n & \ldots & 1/n \\ 1 - 1/n & -1/n & -1/n & \ldots & -1/n \\ 1 & 1 - 1/n & -1/n & \ldots & -1/n \\ \vdots \\ -1/n & -1/n & -1/n & \ldots & 1 - 1/n \end{pmatrix} \begin{pmatrix} Y_1 & Y_2 & Y_3 & \ldots Y_n \end{pmatrix}'$$

$$Z = AY$$

Also $Z \sim N(A\mu, A\Sigma A')$

$$\begin{aligned} A\Sigma A' &= AA' \text{ since } \Sigma = I \\ &= \begin{pmatrix} \frac{n}{n^2} & 0 & 0 & \ldots & 0 \\ 0 & (1 - \frac{1}{n})^2 + \frac{n}{n^2} & -2/n(1 - 1/n) + \frac{n-2}{n} & \ldots & -2/n(1 - 1/n) + \frac{n-2}{n} \\ \vdots \\ 0 & -2/n(1 - 1/n) + \frac{n-2}{n} & -2/n(1 - 1/n) + \frac{n-2}{n} & \ldots & (1 - \frac{1}{n})^2 + \frac{n}{n^2} \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{n} & 0 & 0 & 0 & \ldots & 0 \\ 0 & \frac{1}{n} + (1 - \frac{1}{n})^2 & -\frac{1}{n} & -\frac{1}{n} & \ldots & -\frac{1}{n} \\ \vdots \\ 0 & -\frac{1}{n} & -\frac{1}{n} & -\frac{1}{n} & \ldots & \frac{1}{n} + (1 - \frac{1}{n})^2 \end{pmatrix} \\ &= B \end{aligned}$$

Thus M.g.f. of $Z = AY$ is (using Theorem 2.2 with $d = 0$)

$$
\begin{aligned}
E[\exp(t'AY)] &= \exp(t'A\mu + \frac{1}{2}t'A\Sigma A't) \\
&= \exp(t'A\mu + \frac{1}{2}t'AA't)) \text{ using } \Sigma = I \\
&= \exp(t'A\mu + \frac{1}{2}t'Bt))
\end{aligned}
$$

And hence $Z = \begin{pmatrix}\bar{Y} & Y_1 - \bar{Y} & Y_2 - \bar{Y_3} \ldots Y_n - \bar{Y}\end{pmatrix}'$ follows a multivariate distribution such that $Cov(\bar{Y}, Y_i - \bar{Y}) = 0 \implies \bar{Y}$ and $Y_i - \bar{Y}$ are independent (for all i)

Let's call $X = \begin{pmatrix}Y_1 - \bar{Y} & Y_2 - \bar{Y_3} \ldots Y_n - \bar{Y}\end{pmatrix}'$

Then, from above we have $\bar{Y}$ and X are indepedent (also follows from rom theorem 2.4)

Then,

$$\sum_i (Y_i - \bar{Y})^2 = X'X$$

Since $\bar{Y}, X$ are independent $\implies \bar{Y}, X'X$ are independent