MATH 542 Homework 6

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Problem 1

Problem 1a

To find: $f_{y_2,y_4}(y_1, y_3) = \int_{-\infty}^{\infty} f(y_1, y_2, y_2, y_4)dy_2dy_4$ For marginalising a MVN, we simply drop the irrelevant terms(terms with respect to which marginalisation is performed, as they integrate to 1)

Joint Marginal distribution of y_1, y_3: $f_{y_2,y_4}(y_1, y_3) \sim N(\begin{pmatrix} 1 \\ 3 \end{pmatrix}, \begin{pmatrix} 4 & -1 \\ -1 & 5 \end{pmatrix})$

Problem 1b

$f_{y_1,y_3,y_4}(y_2) \sim N(3, 5)$

Problem 1c

$z = y_1 + 2y_2 - y_3 + 3y_4$

Thus, $z = aY$ where $a = \begin{pmatrix} 1 & 2 & -1 & 3 \end{pmatrix}$ and $Y = \begin{pmatrix} y_1 & y_2 & y_3 & y_4 \end{pmatrix}'$

Thus, $Ez = aE[y] = -4$

$$\begin{aligned} Var(z) &= aVar(y)a' \\ &= 79 \text{ using `R`} \end{aligned}$$

Problem 1d

$z_1 = a_1y$ and $z_2 = a_2y$ where $a_1 = \begin{pmatrix} 1 & 1 & -1 & -1 \end{pmatrix}$ and $a_2 = \begin{pmatrix} -3 & 1 & 2 & -2 \end{pmatrix}$

Then $f_{z_1,z_2} \sim N(\begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}, S)$

$\mu_1 = a_1'E[y] = 2$

$\mu_2 = a_2'E[y] = 9$

$\Sigma_{11}^z = a_1\Sigma a_1^T = 11$

$\Sigma_{22}^z = a_2\Sigma a_2^T = 154$

$\Sigma_{12}^z = \Sigma_{21}^z = a_2\Sigma a_2^T = -6$

Thus, $Z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \sim N(\begin{pmatrix} 2 \\ 9 \end{pmatrix}, \begin{pmatrix} 11 & -6 \\ -6 & 154 \end{pmatrix})$

Problem 1e

$\mu' = \mu_1 + \Sigma_{12}\Sigma_{22}^{-1}(x_2 - \mu_2)$

$Cov = \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}$

$$\mu' = \begin{pmatrix}1\\2\end{pmatrix} + \begin{pmatrix}-1 & 2\\3 & -2\end{pmatrix}\begin{pmatrix}5 & -4\\-4 & 4\end{pmatrix}^{-1}\begin{pmatrix}y_3-3\\y_4+2\end{pmatrix} = \begin{pmatrix}1\\2\end{pmatrix} + \begin{pmatrix}1 & 1.5\\1 & 0.5\end{pmatrix}\begin{pmatrix}y_3-3\\y_4+2\end{pmatrix}$$

$$Cov' = \begin{pmatrix}4 & 2\\2 & 6\end{pmatrix} - \begin{pmatrix}-1 & 2\\3 & -2\end{pmatrix}\begin{pmatrix}5 & -4\\-4 & 4\end{pmatrix}^{-1}\begin{pmatrix}-1 & 3\\2 & -2\end{pmatrix} = \begin{pmatrix}2 & 2\\2 & 4\end{pmatrix}$$

$f(y_1, y_2|y_3, y_4) = N(\mu', Cov)$

Problem 1f

$$\mu' = \begin{pmatrix}1\\3\end{pmatrix} + \begin{pmatrix}-1 & 2\\5 & -4\end{pmatrix}\begin{pmatrix}3 & -2\\-4 & 4\end{pmatrix}^{-1}\begin{pmatrix}y_2-2\\y_4+2\end{pmatrix} = \begin{pmatrix}1\\3\end{pmatrix} + \begin{pmatrix}1 & 1\\1 & -0.5\end{pmatrix}\begin{pmatrix}y_2-2\\y_4+2\end{pmatrix}$$

$$Cov' = \begin{pmatrix}4 & 2\\-1 & 3\end{pmatrix} - \begin{pmatrix}-1 & 2\\5 & -4\end{pmatrix}\begin{pmatrix}3 & -2\\-4 & 4\end{pmatrix}^{-1}\begin{pmatrix}2 & 6\\2 & -2\end{pmatrix}$$

$f(y_1, y_2|y_3, y_4) = N(\mu', Cov)$

Problem 1g

$Cov(y_1, y_3) = -1$

Problem 1h

$$\mu' = 1 - \begin{pmatrix}2 & -1 & 2\end{pmatrix}\begin{pmatrix}6 & 3 & -2\\3 & 5 & -4\\-2 & -4 & 4\end{pmatrix}^{-1}\begin{pmatrix}y_2-2\\y_3-3\\y_4+2\end{pmatrix}$$

$$Cov' = 4 - \begin{pmatrix}2 & -1 & 2\end{pmatrix}\begin{pmatrix}6 & 3 & -2\\3 & 5 & -4\\-2 & -4 & 4\end{pmatrix}^{-1}\begin{pmatrix}2\\-1\\2\end{pmatrix}$$

$f(y_1, y_2|y_3, y_4) = N(\mu', Cov)$

Problem 2

Since $\sigma_{12} = \sigma_{13} = \sigma_{14} = 0$ and y follows a MVN, by Theorem 2.2, y_1 is pairwise independent with y_2, y_3, y_4

Problem 3

$$y_2 - \Sigma_{21}\Sigma_{11}^{-1}y_1 = \begin{pmatrix}0_{n-r\times r} & I_{n-r\times n-r}\end{pmatrix}\begin{pmatrix}y_1\\y_2\end{pmatrix} + \begin{pmatrix}-\Sigma_{21}\Sigma_{11}^{-1} & 0_{n\times r}\end{pmatrix}\begin{pmatrix}y_1\\y_2\end{pmatrix}$$

$$E(y_2 - \Sigma_{21}\Sigma_{11}^{-1}y_1) = \begin{pmatrix}0_{n-r\times r} & I_{n-r\times n-r}\end{pmatrix}\begin{pmatrix}Ey_1\\Ey_2\end{pmatrix} + \begin{pmatrix}-\Sigma_{21}\Sigma_{11}^{-1} & 0_{n\times r}\end{pmatrix}\begin{pmatrix}Ey_1\\Ey_2\end{pmatrix}$$

$$= \mu_2 - \Sigma_{21}\Sigma_{11}^{-1}\mu_1$$

$$
\begin{aligned}
Cov(y_2 - \Sigma_{21}\Sigma_{11}^{-1}y_1) &= Cov(\begin{pmatrix}0_{n-r\times r} & I_{n-r\times n-r}\end{pmatrix}\begin{pmatrix}y_1\\ y_2\end{pmatrix} + \begin{pmatrix}-\Sigma_{21}\Sigma_{11}^{-1} & 0_{n\times r}\end{pmatrix}\begin{pmatrix}y_1\\ y_2\end{pmatrix}) \\
&= Cov(aY + bY) \\
&= (a+b)Var(Y)(a+b)^T \\
&= \begin{pmatrix}-\Sigma_{21}\Sigma_{11}^{-1} & I\end{pmatrix}\begin{pmatrix}\Sigma_{11} & \Sigma_{12}\\ \Sigma_{21} & \Sigma_{22}\end{pmatrix}\begin{pmatrix}-(\Sigma_{11}^{-1})^T\Sigma_{21}^T\\ I^T\end{pmatrix} \\
&= \begin{pmatrix}-\Sigma_{21}+\Sigma_{21} & -\Sigma_{21}\Sigma_{11}^{-1}\Sigma_{12}+\Sigma_{22}\end{pmatrix}\begin{pmatrix}-(\Sigma_{21}\Sigma_{11}^{-1})^T\\ I^T\end{pmatrix} \\
&= \Sigma_{22} - \Sigma_{21}\Sigma_{11}^{-1}\Sigma_{12}
\end{aligned}
$$

Problem 4

Problem 4a

Given $t = \frac{z}{\sqrt{\frac{u}{\rho}}} \sim t(\rho)$ we know the following facts:

- $Z \sim N(0,1)$
- $u \sim \chi^2_\rho$
- Z and u are independent

$t^2 = \frac{z^2}{\frac{u}{\rho}} \sim \frac{\chi^2_1}{\chi^2_\rho} \sim F(1,\rho)$

Problem 5.3

We consider first the following vector: $Z = \begin{pmatrix}\bar{Y} & Y_1 - Y_2 & Y_2 - Y_3 \dots Y_n - Y_{n-1}\end{pmatrix}'$

Let's call $X = \begin{pmatrix}Y_1 - Y_2 & Y_2 - Y_3 \dots Y_{n-1} - Y_n\end{pmatrix}'$ so that it allows us to write $\sum_{i=1}^{n-1}(Y_i - Y_{i+1})^2 = X'X$

Now $Z = \begin{pmatrix}\bar{Y} & X\end{pmatrix}$

$$
\begin{pmatrix}\bar{Y} & Y_1 - Y_2 & Y_2 - Y_3 \dots Y_{n-1} - Y_n\end{pmatrix}' = \begin{pmatrix}1/n & 1/n & 1/n & \dots & 1/n\\ 1 & -1 & 0 & \dots & 0\\ 0 & 1 & -1 & \dots & 0\\ \vdots & & & & \\ 0 & 0 & 0 & \dots & -1\end{pmatrix}\begin{pmatrix}Y_1 & Y_2 & Y_3 & \dots Y_n\end{pmatrix}'
$$

$$
Z = AY
$$

Also $Z \sim N(A\mu, A\Sigma A')$
$A\mu = \begin{pmatrix}\mu & 0 & 0 \dots 0\end{pmatrix}$
$A\Sigma A' = AA'$ since $\Sigma = I$

$$
AA' = \begin{pmatrix}1/n & 0 & 0 & \dots & 0\\ 0 & 2 & -1 & \dots & 0\\ 0 & -1 & 2 & -1 & \dots 0\\ \vdots 0 & 0 & 0 & 0 & \dots 2\end{pmatrix}
$$

Thus, $Z = \begin{pmatrix}\bar{Y} & X_{n\times 1}\end{pmatrix}'$ is a MVN such that $\bar{Y}$ and X are independent (since the covariance is 0)

We also know that $\sum_{i=1}^{n-1}(Y_i - Y_{i+1})^2 = X'X = h(X)$

Since, functions of independent random variables are also independent $\bar{Y}$ and $h(X) = X'X$ are independent.

Problem 5.11

$$Z = \begin{pmatrix} \phi & 1 & 0 & 0 & \dots & 0 \\ 0 & \phi & 1 & 0 & \dots & 0 \\ 0 & 0 & \phi & 1 & \dots & 0 \\ \vdots \\ 0 & 0 & 0 & 0 & \dots & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_n \end{pmatrix} = AY$$

Since $Y \sim N(0, \sigma^2 I) \implies Z \sim N(0, \sigma^2 AA^T)$

$$\text{where } AA^T = \begin{pmatrix} \phi^2+1 & \phi & 0 & 0 & \dots & 0 \\ \phi & 1+\phi^2 & \phi & 0 & \dots & 0 \\ 0 & \phi & 1+\phi^2 & \phi & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \\ 0 & 0 & 0 & \dots & 1+\phi^2 & \phi \end{pmatrix}$$