MATH 542 Homework 7

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\begin{aligned}
Y_i &= \beta_0 + \beta_1 X_i + \epsilon_i \\
\epsilon_i &= Y_i - \beta_0 - \beta_1 X_i \\
\epsilon_i^2 &= (Y_i - \beta_0 - \beta_1 X_i)^2 \\
Q &= \sum_i \epsilon_i^2 \sum_i (Y_i - \beta_0 - \beta_1 X_i)^2
\end{aligned}
$$

We want to minimize the residual $\epsilon_i = Y_i - \beta_0 - \beta_1 X_i = Y_i - \hat{Y}_i$ so $\frac{\partial \sum_i \epsilon_i^2}{\partial \beta_0} = \frac{\partial \sum_i \epsilon_i^2}{\partial \beta_1} = 0$

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\begin{aligned}
\frac{\partial Q}{\partial \beta_0} &= -2 \sum_i (Y_i - \beta_0 - \beta_1 X_i) = 0 \\
\hat{\beta_0} &= \bar{Y} - \hat{\beta_1} \bar{X}
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial Q}{\partial \beta_1} &= -2 \sum_i X_i (Y_i - \beta_0 - \beta_1 X_i) = 0 \\
\sum_i X_i (Y_i - \hat{\beta_0} - \hat{\beta_1} X_i) = 0 \\
\sum_i X_i (Y_i - \bar{Y} + \hat{\beta_1} \bar{X} - \hat{\beta_1} X_i) = 0 \\
\hat{\beta_1} &= \frac{\sum_i X_i (Y_i - \bar{Y})}{\sum_i X_i (X_i - \bar{X})}
\end{aligned}
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Using $\sum \bar{X}(X_i - \bar{X}) = 0 \implies \sum X_i (X_i - \bar{X}) = \sum (X_i - \bar{X})(X_i - \bar{X})$ and using $\sum \bar{X}(Y_i - \bar{Y}) = 0$ we get $\sum_i X_i (Y_i - \bar{Y}) = \sum_i (X_i - \bar{X})(Y_i - \bar{Y})$

and hence $\hat{\beta_1} = \frac{\sum_i (X_i - \bar{X})(Y_i - \bar{Y})}{\sum_i (X_i - \bar{X})^2}$

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\begin{aligned}
E[\hat{\beta_0}] &= E[\bar{Y}] - \bar{X} E[\hat{\beta_1}] \\
E[\bar{Y}] &= E[\beta_0 + \beta_1 \bar{X} + 1/n \sum_i \epsilon_i] \\
&= \beta_0 + \beta_1 \bar{X}
\end{aligned}
$$

Now consider $E[\hat{\beta}_1] = \frac{\sum_i (X_i - \bar{X})(EY_i - E\bar{Y})}{\sum_i (X_i - \bar{X})^2} = \frac{\sum_i (X_i - \bar{X})(\beta_1 X_i - \beta_1 \bar{X})}{\sum_i (X_i - \bar{X})^2} = \beta_1$ using which we get:

$$E[\hat{\beta}_0] = E[\bar{Y}] - \bar{X}E[\hat{\beta}_1] = \beta_0 + \beta_1\bar{X} - \beta_1\bar{X} = \beta_0$$

Problem 2

$$\begin{aligned}
Var(\hat{\beta}_1) &= \frac{1}{(\sum(X_i - \bar{X})^2)^2} Var(\sum_i (X_i - \bar{X})(Y_i - \bar{Y})) \\
&= \frac{1}{(\sum(X_i - \bar{X})^2)^2} Var(\sum_i (X_i - \bar{X})(Y_i)) \\
&= \frac{1}{(\sum(X_i - \bar{X})^2)^2} \sum_i (X_i - \bar{X})^2 Var(Y_i) \\
&= \frac{1}{(\sum(X_i - \bar{X})^2)^2} \sum_i (X_i - \bar{X})^2 \sigma^2 \\
&= \frac{\sigma^2}{(\sum(X_i - \bar{X})^2)}
\end{aligned}$$

$$\begin{aligned}
\hat{\beta}_0 &= \bar{Y} - \hat{\beta}_1\bar{X} \\
Var(\hat{\beta}_0) &= Var(\bar{Y}) + \bar{X}^2 Var(\hat{\beta}_1) \\
&= \frac{\sigma^2}{n} + \bar{X}^2 (\frac{\sigma^2}{(\sum(X_i - \bar{X})^2)})
\end{aligned}$$

Problem 3

$$\begin{aligned}
Y_i &= \beta_0 + \beta_1 X_i + \epsilon_i \\
\hat{Y}_i &= \beta_0 + \beta_1 X_i \\
\frac{\partial Q}{\partial \beta_0} &= -2\sum_i (Y_i - \beta_0 - \beta_1 X_i) = -2\sum_i (Y_i - \hat{Y}_i) = 0 \\
\frac{\partial Q}{\partial \beta_1} &= -2\sum_i X_i(Y_i - \beta_0 - \beta_1 X_i) = 0 \\
&= \sum_i (\hat{Y}_i/\beta_1 - \beta_0/\beta_1)(Y_i - \hat{Y}_i) = 0 \\
&\implies \sum_i (\hat{Y}_i)(Y_i - \hat{Y}_i) = 0
\end{aligned}$$

Now,

$$
\begin{aligned}
\sum_i (Y_i - \bar{Y})^2 &= \sum_(Y_i - \hat{Y}_i + \hat{Y}_i - \bar{Y})^2 \\
&= \sum (Y_i - \hat{Y}_i)^2 + \sum_i (\hat{Y}_i - \bar{Y})^2 + 2\sum_i (Y_i - \hat{Y}_i)(\hat{Y}_i - \bar{Y}) \\
&= \sum (Y_i - \hat{Y}_i)^2 + \sum_i (\hat{Y}_i - \bar{Y})^2 + 2\sum_i \hat{Y}_i(Y_i - \hat{Y}_i) - 2\sum_i \bar{Y}(Y_i - \hat{Y}_i) \\
&= \sum (Y_i - \hat{Y}_i)^2 + \sum_i (\hat{Y}_i - \bar{Y})^2
\end{aligned}
$$

where the two last terms are zero using the properties derived previous to the last set of equations

Problem 4

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\begin{aligned}
r &= \frac{\sum_i (x_i - \bar{x})(y_i - \bar{y})}{\sqrt{\sum_i (x_i - \bar{x})^2 \sum_i (y_i - \bar{y})^2}} \\
r^2 &= \frac{(\sum_i (x_i - \bar{x})(y_i - \bar{y}))^2}{\sum_i (x_i - \bar{x})^2 \sum_i (y_i - \bar{y})^2} = \frac{s_{xy}^2}{s_{xx} s_{yy}} \\
R^2 &= 1 - \frac{SSE}{SST} = \frac{SSR}{SST} \\
SSE(rror) &= \sum_i (y_i - \hat{y}_i)^2 \\
SST(otal) &= \sum_i (y_i - \bar{y})^2 \\
SSR(egression) &= \sum_i (\hat{y}_i - \bar{y})^2
\end{aligned}
$$

Consider $y = a + bx$ then from the first problem we have: $b = \frac{s_{xy}}{s_{xx}}$

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\begin{aligned}
SSR &= \sum_i (\hat{y}_i - \bar{y})^2 \\
&= \sum_i (a + bx_i - \bar{y})^2 \\
&= \sum (\bar{y} - b\bar{x} + bx_i - \bar{y})^2 \\
&= b^2 s_{xx} \\
&= \frac{s_{xy}^2}{s_{xx}^2} s_{xx} \\
&= \frac{s_{xy}^2}{s_{xx}}
\end{aligned}
$$

$$
\begin{aligned}
R^2 &= \frac{SSR}{SST} \\
&= \frac{\frac{s_{xy}^2}{s_{xx}}}{s_{yy}} \\
&= \frac{s_{xy}^2}{s_{xx}s_{yy}} \\
&= r^2
\end{aligned}
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Problem 5

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\begin{aligned}
L(\sigma^2, \beta_0, \beta_1) &= (2\pi\sigma^2)^{-\frac{n}{2}} \prod_i exp(-\frac{1}{2\sigma^2}(y_i - \beta_0 - \beta_1 x_i)^2) \\
\log(L) &= -\frac{n}{2}\log(2\pi) - n\log\sigma^2 - \frac{1}{2\sigma^2}\sum_i (y_i - \beta_0 - \beta_1 x_i)^2 \\
\frac{\partial \log L}{\partial \beta_0}|^{\beta_{MLE}} &= -\frac{-2}{2\sigma^2}\sum_i (y_i - \beta_0 - \beta_1 x_i) \\
&\implies \beta_0^{MLE} = \bar{Y} - \beta_1^{MLE}\bar{X} \\
\frac{\partial \log L}{\partial \beta_1}|^{\beta_{MLE}} &= -\frac{-2}{2\sigma^2}\sum_i x_i(y_i - \beta_0 - \beta_1 x_i) \\
&\implies \beta_1^{MLE} = \frac{\sum_i (x_i - \bar{x})(y_i - \bar{y})}{\sum_i (x_i - \bar{x})^2} \\
\frac{\partial \log L}{\partial \sigma^2}|^{\beta_{MLE}} &= -\frac{n}{2\sigma^2} + \frac{1}{2\sigma^4}\sum_i (y_i - \beta_0 - \beta_1 x_1)^2 \\
&\implies \sigma^2_{MLE} = \frac{1}{n}\sum_i (y_i - \beta_0^{MLE} - \beta_1^{MLE} x_1)^2
\end{aligned}
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