MATH 542 Homework 8

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Problem 3a.2

Consider $||Y - X\beta||^2$. Since X is full rank $\hat{\beta} = (X'X)^{-1}X'Y$ This involves $\frac{||Y-X\beta||^2}{\partial\beta_i} = 0$ Also $\hat{Y}_i = \beta_0 + x_{i1}\beta_1 + \cdots + x_{ip-1}\beta_{p-1}$ For $i = 0$:

$$\begin{aligned}
\frac{||Y - X\beta||^2}{\partial\beta_i} &= 0 \\
\frac{\sum(Y_i - (\beta_0 + x_{i1}\beta_1 + \cdots + x_{ip-1}\beta_{p-1}))^2}{\partial\beta_0} &= 0 \\
\sum(Y_i - (\beta_0 + x_{i1}\beta_1 + \cdots + x_{ip-1}\beta_{p-1})) &= 0 \\
\sum(Y_i - \hat{Y}_i) &= 0
\end{aligned}$$

Problem 3a.3

$$\begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 2 & -1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} \theta \\ \phi \end{pmatrix}$$

Thus $X = \begin{pmatrix} 1 & 0 \\ 2 & -1 \\ 1 & 22 \end{pmatrix}$ $\beta = X'X$ and hence using R:

$$\begin{aligned}
\theta &= 0.167Y_1 + 0.333Y_2 + 0.167Y_3 \\
\phi &= -0.2Y_2 + 0.4Y_3
\end{aligned}$$

Problem 3a.4

$$\begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \end{pmatrix} = \begin{pmatrix} 1 & -1 & 1 \\ 1 & 0 & -2 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \end{pmatrix}$$

$$\begin{pmatrix}\beta_0\\\beta_1\\\beta_2\end{pmatrix} = (X'X)^{-1}X'Y$$

$$\begin{pmatrix}\beta_0\\\beta_1\\\beta_2\end{pmatrix} = \begin{pmatrix}\frac{1}{3} & \frac{1}{3} & \frac{1}{3}\\ -0.5 & 0 & 0.5\\ \frac{1}{6} & -\frac{1}{3} & \frac{1}{6}\end{pmatrix}\begin{pmatrix}Y_1\\Y_2\\Y_3\end{pmatrix}$$

Consider $\beta_2 = 0$:

$$\begin{pmatrix}Y_1\\Y_2\\Y_3\end{pmatrix} = \begin{pmatrix}1 & -1\\ 1 & 0\\ 1 & 1\end{pmatrix}\begin{pmatrix}\beta_0\\\beta_1\end{pmatrix}$$

Then using R:

$$\begin{pmatrix}\beta_0\\\beta_1\end{pmatrix} = (X'X)^{-1}X'Y$$

$$\begin{pmatrix}\beta_0\\\beta_1\end{pmatrix} = \begin{pmatrix}\frac{1}{3} & \frac{1}{3} & \frac{1}{3}\\ -0.5 & 0 & 0.5\end{pmatrix}\begin{pmatrix}Y_1\\Y_2\\Y_3\end{pmatrix}$$

Problem 3a.7

$\hat{Y} = X\hat{\beta} = PY$

$$\begin{aligned}\sum \hat{Y}_i(Y_i - \hat{Y}_i) &= \hat{Y}'(Y - \hat{Y}'\\ &= Y'P'(Y - PY)\\ &= Y'P(I_n - P)Y\\ &= Y'(P - P^2)Y\\ &= 0 \text{ Since P is idempotent}\end{aligned}$$

Problem 3b.3

Consider $\bar{Y} = \frac{\sum_i Y_i}{n}$

$E[\bar{Y}] = \theta$ so $\bar{Y}$ is unbiased estimate.

Also using Rao's minimum variance lower bound, $\alpha'\beta$ is a minimum variance estimate for $\mathcal{N}(X\beta, \sigma^2)$, Thus $\bar{Y}$ is both unbiased and minimum variance.

Problem 3b.4

$$Y_i = \beta_0 + \beta_1(x_{i1} - \bar{x_1}) + \beta_2(x_{i2} - \bar{x_2}) + \epsilon_i$$

$$\begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \\ \vdots \\ Y_n \end{pmatrix} = \begin{pmatrix} 1 & x_{11} - \bar{x_1} & x_{12} - \bar{x_2} \\ 1 & x_{21} - \bar{x_1} & x_{22} - \bar{x_2} \\ \vdots & \vdots & \vdots \\ 1 & x_{n1} - \bar{x_1} & x_{n2} - \bar{x_2} \end{pmatrix} \begin{pmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \end{pmatrix}$$

$$X = \begin{pmatrix} 1 & x_{11} - \bar{x_1} & x_{12} - \bar{x_2} \\ 1 & x_{21} - \bar{x_1} & x_{22} - \bar{x_2} \\ \vdots & \vdots & \vdots \\ 1 & x_{n1} - \bar{x_1} & x_{n2} - \bar{x_2} \end{pmatrix}$$

$$X'X = \begin{pmatrix} n & \sum x_{i1} - n\bar{x_1} & \sum x_{i2} - n\bar{x_2} \\ \sum x_{i1} - n\bar{x_1} & \sum (x_{i1} - \bar{x_1})^2 & \sum (x_{i1} - \bar{x_1})(x_{i2} - \bar{x_2}) \\ \sum x_{i2} - n\bar{x_2} & \sum (x_{i1} - \bar{x_1})(x_{i2} - \bar{x_2}) & \sum (x_{i2} - \bar{x_2})^2 \end{pmatrix}$$

$$= \begin{pmatrix} n & 0 & 0 \\ 0 & \sigma_1^2 & r\sigma_1\sigma_2 \\ 0 & r\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix}$$

$$(X'X)^{-1} = \frac{1}{\sigma_1^2\sigma_2^2(1-r^2)} \begin{pmatrix} \frac{1}{n(\sigma_1^2\sigma_2^2(1-r^2))} & 0 & 0 \\ 0 & \sigma_2^2 & r\sigma_1\sigma_2 \\ 0 & r\sigma_1\sigma_2 & \sigma_1^2 \end{pmatrix}$$

Thus $Var(\beta_1) = \sigma^2(X'X)^{-1} = \frac{\sigma^2\sigma_2^2}{\sigma_1^2\sigma_2^2(1-r^2)} = \frac{\sigma^2}{\sigma_1^2(1-r^2)}$