MATH 542 Homework 9

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Problem 1

Since $\mathbf{X}$ is full rank, the least sqare estimate of β for the linear model $\mathbf{Y} = \mathbf{y}\beta + \epsilon$ then $\hat{\beta} = (\mathbf{X'X})^{-1}\mathbf{X'y}$ also this is an unbiased estimator of β that is $E[\hat{\beta}] = \beta$

$$\begin{aligned}
a'\beta &= a'E[\hat{\beta}] \\
&= a'E[(\mathbf{X'X})^{-1}\mathbf{X'y}] \\
&= E[a'(\mathbf{X'X})^{-1}\mathbf{X'y}] \\
&= E[b'\mathbf{y}]
\end{aligned}$$

Hence when $\mathbf{X}$ is full rank, we have can take $b' = a'(\mathbf{X'X})^{-1}\mathbf{X'}$ making every $a'\beta$ estimable

For each individual β_i to be estimable, we set $a' = \begin{pmatrix} 0 & 0 & \dots & 0 & 1 & 0\dots 0 \end{pmatrix}$ setting $a_i = 1$ remaining zero.

Problem 2

Since each $a_i'\beta$ is estimable: $a_i'\beta = E[b_i'Y]$ Consider $\lambda_i \in \mathcal{R}$

$$\begin{aligned}
\lambda_1 a_1'\beta + \lambda_2 a_2'\beta + \dots + \lambda_n a_n'\beta &= \lambda_1 E[b_1'Y] + \lambda_2 E[b_2'Y] + \dots \lambda_n E[b_n'Y] \\
&= \sum_i E[\lambda_i b_i' Y] \\
&= \sum E[c'Y] \text{ where } c' = \lambda \mathbf{Ib'}
\end{aligned}$$

where $\lambda = \begin{pmatrix} \lambda_1 & \lambda_2 & \dots & \lambda_n \end{pmatrix}$ and $b' = \begin{pmatrix} b_1 & b_2 & \dots b_n \end{pmatrix}$

Hence linear combination of $a_i'\beta$ is also estimable.

Problem 3

$$X = \begin{pmatrix} 1 & 1 & 3 \\ 1 & 1 & -2 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \beta \qquad = \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix}$$

$$\begin{aligned}
\beta_1 &= \begin{pmatrix} 1 & 1 & 0 \end{pmatrix} \beta \\
\beta_1 - \beta_2 &= \begin{pmatrix} 1 & -1 & 0 \end{pmatrix} \beta \\
5\beta_1 + 3\beta_2 + 9\beta_3 &= \begin{pmatrix} 5 & 3 & 9 \end{pmatrix} \beta
\end{aligned}$$

X has full column rank and hence we can make use of the theorem we proved in Problem 1 to say that all three cases (a, b, c) are indeed estimable, that is since X is full rank, every $a'\beta$ is estimable (One particular case of the corollary proved in Problem 1 here is the case a where $a' = \begin{pmatrix}1 & 0 & 0\end{pmatrix}$

Problem 4

Problem 4.a

Define $\mathbf{y} = \begin{pmatrix}Y_1 & Y_2 & Y_3\end{pmatrix}'$, $\tau = \begin{pmatrix}\tau_1 & \tau_2 & \tau_3\end{pmatrix}$, $\epsilon = \begin{pmatrix}\epsilon_1 & \epsilon_2 & \epsilon_3\end{pmatrix}$

Thus, $\mathbf{y} = \begin{pmatrix}1 & 1 & 1\\ 1 & 0 & 1\\ 0 & 1 & 0\end{pmatrix}\tau + \epsilon$

Rank of design matrix is 2

Problem 4.b

$E[b'Y] = a'\beta = b'X\beta$ iff $a' = b'X$ or $a = Xb'$ τ_2 is estimable:

$$\begin{aligned}\tau_2 = a'\beta &= \begin{pmatrix}0 & 1 & 0\end{pmatrix}\tau\\ &= EY_3\\ &= E[\begin{pmatrix}0 & 0 & 1\end{pmatrix}]\end{aligned}$$

$\tau_1 = \begin{pmatrix}1 & 0 & 0\end{pmatrix}\tau$ so we need to find b such that $\begin{pmatrix}1 & 0 & 0\end{pmatrix}' = X'b$ It is clear to see no b does not exist.

$\tau_2 = \begin{pmatrix}0 & 1 & 0\end{pmatrix}\tau$ is estimable because $b = \begin{pmatrix}0 & 1 & 0\end{pmatrix}$

τ_3 is not estimable because of symmetry with τ_1

Problem 4.c

$\tau_1 - 2\tau_2 + \tau_3 = \begin{pmatrix}1 & -2 & 1\end{pmatrix}\tau = \begin{pmatrix}1 & 1 & 0\\ 1 & 0 & 1\\ 0 & 1 & 0\end{pmatrix}\begin{pmatrix}0\\ 1\\ -2\end{pmatrix}$ and hence it is estimable

Problem 4.d

A possible unbiasd estimator of $\tau_1 - 2\tau_2 + \tau_3 = E[Y_2 - 2Y_3]$ i.e $Y_2 - 2Y_3$ whihc is not necessarily BLUE.

For BLUE, we simply take the OLS estimate of β as $\hat{\beta} = \mathbf{X'X}^{-1}\mathbf{X'y}$ and from Gauss-Markov model BLUE follows.

Generalized inverse(using R) $\mathbf{X'X}^{-1} = \begin{pmatrix}\frac{1}{6} & \frac{-1}{6} & \frac{1}{6}\\ \frac{-1}{6} & \frac{2}{3} & \frac{-1}{6}\\ \frac{1}{6} & \frac{-1}{6} & \frac{1}{6}\end{pmatrix}$

so $\hat{\beta} = \mathbf{X'X}^{-1}\mathbf{X'y} = \begin{pmatrix}\frac{1}{6} & \frac{1}{3} & \frac{-1}{6}\\ \frac{1}{3} & \frac{-1}{3} & \frac{2}{3}\\ \frac{1}{6} & \frac{1}{3} & \frac{-1}{6}\end{pmatrix}\mathbf{y}$

and hence $\tau_1 - 2\tau_2 + \tau_3 = \begin{pmatrix}1 & -2 & 1\end{pmatrix}\begin{pmatrix}\frac{1}{6} & \frac{1}{3} & \frac{-1}{6}\\ \frac{1}{3} & \frac{-1}{3} & \frac{2}{3}\\ \frac{1}{6} & \frac{1}{3} & \frac{-1}{6}\end{pmatrix}\mathbf{y} = \frac{-Y_1+4Y_2-5Y_3}{3}$

Thus BLUE of $\tau_1 - 2\tau_2 + \tau_3$: $\frac{-Y_1+4Y_2-5Y_3}{3}$