MATH 542 Homework 10

Saket Choudhary
skchoudh@usc.edu

March 20, 2016

Problem 3c.1

Problem 3c.1.a

$$
\begin{aligned}
var[S^2] &= Var[\frac{Y'(I_n - P)Y}{n-p}] \\
&= \frac{1}{(n-p)^2} Var[Y'(I_n - P)Y] \\
&= \frac{1}{(n-p)^2} \times (2\sigma^4(n-p)) \\
&= \frac{2\sigma^4}{n-p}
\end{aligned}
$$

Problem 3c.1.b

$$
\begin{aligned}
A_1 &= \frac{1}{n-p+2}[I_n - X(X'X)^{-1}X'] \\
&= \frac{R}{n-p+2} \\
E[(Y'A_1Y - \sigma^2)^2] &= Var(Y'A_1Y - \sigma^2) + (E[Y'A_1Y - \sigma^2])^2 \\
&= Var(Y'A_1Y) + (E[Y'A_1Y] - \sigma^2)^2 \\
&= \frac{Var(Y'RY)}{(n-p+2)^2} + (\frac{E[Y'RY]}{(n-p+2)} - \sigma^2)^2 \\
&= \frac{2\sigma^4(n-p)}{(n-p+2)^2} + (\frac{\sigma^2(n-p)}{n-p+2} - \sigma^2)^2 \text{ using 3.12 from textbook} \\
&= \frac{2\sigma^4(n-p)}{(n-p+2)^2} + \frac{4\sigma^4}{(n-p+2)^2} \\
&= \frac{2\sigma^4}{n-p+2}
\end{aligned}
$$

Problem 3c.1.c

$$\begin{aligned}
E[Y'A_1Y] &= \frac{E[Y'RY]}{n-p+2} \\
&= \frac{\sigma^2(n-p)}{n-p+2} \text{ using 3.12 from textbook} \\
MSE[Y'A_1Y] &= E[(Y'A_1Y-\sigma^2)^2] \\
&= \frac{2\sigma^4}{n-p+2} \\
MSE[S^2] &= E[S^2-(E[S^2])^2] \\
&= Var(S^2) \\
&= \frac{2\sigma^4}{n-p} \\
&< \frac{2\sigma^4}{n-p+2} \\
&\leq MSE[Y'A_1Y]
\end{aligned}$$

Problem 3d.1

Problem 3d.1.a

Given $Y_i \sim N(\theta, \sigma^2)$ or $Y_i = \theta + \epsilon_i$ where $\epsilon_i \sim N(0, \sigma^2)$
$\mathbf{Y} = \mathbf{1_n}\theta + \epsilon$ thus $\hat{\theta} = (\mathbf{1_n}'\mathbf{1_n})^{-1}\mathbf{1_n'}Y = \frac{1}{n}\mathbf{1_n'}Y = \bar{Y}$

Thus, using theorem 3.5(ii) $\bar{Y}$ and $S^2 = \sum_i(Y_i - \bar{Y})^2$ are independent

Problem 3d.1.b

Borrowing from part (a) we have: $RSS = Q = \sum_i(Y_i - \bar{Y})^2 \implies$ using theorem 3.5(iii):

$RSS/\sigma^2 \sim \chi^2_{n-1}$

Problem 3d.2

$$\begin{aligned}
RSS &= Y'(I_n-P)Y \\
&= Y'(I_n-P)Y - \beta'X'(I-P)(Y-X\beta) + Y'(I-P)(-X\beta) \text{ both terms are zero using PX=P and P=} \\
&= (Y-X\beta)'(I_n-P)(Y-X\beta) \\
&= \epsilon'(I_n-P)\epsilon
\end{aligned}$$

$$\begin{aligned}
(\hat{\beta}-\beta)'X'X(\hat{\beta}-\beta) &= Z'Z \\
Z &= X(\hat{\beta}-\beta) \\
&= X((X'X)^{-1}X'Y - (X'X)^{-1}X'X\beta) \\
&= P(Y-X\beta) \\
&= P\epsilon \\
(\hat{\beta}-\beta)'X'X(\hat{\beta}-\beta) &= \epsilon'P'P\epsilon
\end{aligned}$$

$$
\begin{aligned}
Cov[RSS, (\hat{\beta}-\beta)'X'X(\hat{\beta}-\beta)] &= Cov[\epsilon'(I_n - P)\epsilon, \epsilon'P'P\epsilon] \\
&= Cov[\epsilon'(I_n - P)\epsilon, \epsilon'PP\epsilon] \text{ using } P' = P \\
&= Cov[\epsilon'(I_n - P)\epsilon, \epsilon'P\epsilon] \text{ using } PP = P \\
&= \sigma^2(I-P)P \\
&= 0
\end{aligned}
$$

Thus, RSS and $(\hat{\beta}-\beta)'X'X(\hat{\beta}-\beta)$ are indepedent

Problem 3.12

$$
\begin{aligned}
Y &= X\beta + \epsilon \\
\bar{Y} &= \frac{1}{n}\mathbf{1_n}Y \\
\sum_i (Y_i - \hat{Y}_i)^2 &= (Y - X\hat{\beta})'(Y - X\hat{\beta}) \\
&= (Y - X(X'X)^{-1}X'Y)'(Y - X(X'X)^{-1}X'Y) \\
&= (Y - PY)'(Y - PY) \\
&= Y'(I-P)'(I-P)Y \\
&= Y'(I-P)Y \text{ using idempotency of } I - P
\end{aligned}
$$

$$
\begin{aligned}
Cov[\frac{1}{n}\mathbf{1_n}Y, (I-P)Y]& \\
&= \frac{1}{n}\mathbf{1_n}Cov[Y](I-P)' \\
&= \sigma^2(n-p)\frac{1}{n}\mathbf{1_n}(I-P)'
\end{aligned}
$$

Since the first column of the design matrix is all 1, 1_n belongs to the column space of X and is orthogonal to $(I-P)'$ (P being the projection matrix) $\implies Cov[\frac{1}{n}\mathbf{1_n}Y, (I-P)Y] = 0$