MATH 542 Homework 11

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Problem 4b.1

$$F = \frac{n-p}{q}\frac{(Y-c1_n)'(P-P_H)(Y-c1_n)}{(Y-c1_n)'(I_n-P)(Y-c1_n)}$$

Also, $X = \begin{pmatrix} 1 & x_{1,1} & x_{1,2} & \dots & x_{1,p-1} \\ 1 & x_{2,1} & x_{2,2} & \dots & x_{2,p-1} \\ \vdots & \vdots & \vdots & & \\ 1 & x_{n,1} & x_{n,2} & \dots & x_{n,p-1} \end{pmatrix}$

Thus, $1_n \in \mathcal{C}(X)$ and hence $(I-P)1_n = 1_n'(I-P) = (P-P_H)1_n = 1_n'(P-P_H) = 0$ and hence $(Y-c1_n)'(P-P_H)(Y-c1_n) = Y'(P-P_H)Y - c1_n'(P-P_H)Y + cY'(P-P_H)1_n + c^2 1_n'(P-P_H)1_n = Y'(P-P_H)Y$

Similarly $(Y-c1_n)'(I-P)(Y-c1_n) = Y'(I-P)Y$ and hence F statistic is the same as $F = \frac{n-p}{q}\frac{Y'(P-P_H)Y}{Y'(I_n-P)Y}$

Problem 4b.4

$$\begin{aligned} \begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \end{pmatrix} &= \begin{pmatrix} 1 & 1 \\ 0 & 2 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix} + \begin{pmatrix} \epsilon_1 \\ \epsilon_2 \end{pmatrix} \\ &= X\beta + \epsilon \\ H:\ \theta_1 &= 2\theta_2 \\ \implies \begin{pmatrix} 1 & -2 \end{pmatrix} \\ &= A\beta = 0 \\ F &= \frac{\frac{RSS_H - RSS}{q}}{\frac{RSS}{n-p}} \end{aligned}$$

Now,

$$\hat{\beta} = (X'X)^{-1}X'Y$$

$$X'X = \begin{pmatrix} 2 & 0 \\ 0 & 6 \end{pmatrix}$$

$$(X'X)^{-1} = \begin{pmatrix} 1/2 & 0 \\ 0 & 1/6 \end{pmatrix}$$

$$\hat{\beta} = \begin{pmatrix} 1/2 & 0 \\ 0 & 1/6 \end{pmatrix} \begin{pmatrix} \frac{Y_1 - Y_3}{2} \\ \frac{Y_1 + 2Y_2 + Y_3}{6} \end{pmatrix}$$

$$\begin{aligned} \hat{Y} &= X\hat{\beta} \\ &= \begin{pmatrix} 1 & 1 \\ 0 & 2 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} \frac{3Y_1 - 3Y_3}{6} \\ \frac{Y_1 + 2Y_2 + Y_3}{6} \end{pmatrix} \\ &= \begin{pmatrix} \frac{4Y_1 + 2Y_2 - 2Y_3}{6} \\ \frac{Y_1 + 2Y_2 + Y_3}{3} \\ \frac{-2Y_1 + 2Y_2 + 4Y_3}{6} \end{pmatrix} \end{aligned}$$

$$\begin{aligned} RSS &= (Y - \hat{Y})'(Y - \hat{Y}) \\ &= \begin{pmatrix} \frac{2Y_1 - 2Y_2 + 2Y_3}{6} & \frac{-Y_1 + Y_2 - Y_3}{3} & \frac{2Y_1 - 2Y_2 + 2Y_3}{6} \end{pmatrix}' \cdot \begin{pmatrix} \frac{2Y_1 - 2Y_2 + 2Y_3}{6} \\ \frac{4y_1 - 4Y_2 - 2Y_3}{6} \\ \frac{2Y_1 - 2Y_2 + 2Y_3}{6} \end{pmatrix} \\ &= \frac{1}{9}(Y_1 - Y_2 + Y_3)^2 + \frac{1}{9}(-Y_1 + Y_2 - Y_3)^2 + \frac{1}{9}(Y_1 - Y_2 + Y_3)^2 \\ &= \frac{1}{3}(Y_1 - Y_2 + Y_3)^2 \end{aligned}$$

$$\begin{aligned} A(X'X)^{-1}A' &= \begin{pmatrix} 1/2 & -1/3 \end{pmatrix} \begin{pmatrix} 1 \\ -2 \end{pmatrix} \\ &= \frac{7}{6} \\ RSS_H - RSS &= [A\hat{\beta}]'[A(X'X)^{-1}A']^{-1}[A'\hat{\beta}] \\ &= \frac{6}{7}(\hat{\beta}_1 - 2\hat{\beta}_2)^2 \end{aligned}$$

Thus,

$$F = \frac{\frac{6}{7}(\hat{\beta}_1 - 2\hat{\beta}_2)^2}{\frac{1}{3}(Y_1 - Y_2 + Y_3)^2}$$

Problem 4b.5

$Y = I\theta + \epsilon$

$H : \theta_1 = \theta_3$ or $\begin{pmatrix} 1 & 0 & -1 & 0 \end{pmatrix} \theta = 0$

$$\begin{aligned}
A &= \begin{pmatrix} 1 & 0 & -1 & 0 \end{pmatrix} \\
\hat{\beta} &= \begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \\ Y_4 \end{pmatrix} \\
A\beta &= \begin{pmatrix} 1 & 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \\ Y_4 \end{pmatrix} \\
&= Y_1 - Y_3 \\
RSS_H - RSS &= \begin{pmatrix} Y_1 & -Y_3 \end{pmatrix} \frac{1}{2} \begin{pmatrix} Y_1 \\ -Y_3 \end{pmatrix} \\
&= \frac{1}{2}(Y_1 - Y_3)^2 \\
RSS = 3S^2 \\
&= 3(Y_1 + Y_2 + Y_3 + Y_4 - 0)^2 \\
F &= \frac{RSS_H - RSS}{3S^2} \frac{3}{1} \\
&= \frac{(Y_1 - Y_3)^2}{2(Y_1 + Y_2 + Y_3 + Y_4)^2}
\end{aligned}$$

Problem 4MISC.2

$$\begin{pmatrix} Y_{11} \\ Y_{12} \\ \vdots \\ Y_{1n} \\ Y_{21} \\ Y_{22} \vdots \\ Y_{2n} \end{pmatrix} = \begin{pmatrix} x_1 & 0 \\ x_2 & 0 \\ \vdots & \\ x_n & 0 \\ 0 & x_1 \\ 0 & x_2 \\ \vdots & \\ 0 & x_n \end{pmatrix} \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} + \epsilon$$

$$\begin{aligned}
H : A\hat{\beta} = 0 \implies \begin{pmatrix} 1 & -1 \end{pmatrix} &= 0 \\
A &= \begin{pmatrix} 1 & -1 \end{pmatrix} \\
X'X^{-1} &= \begin{pmatrix} \frac{1}{\sum_i x_i^2} & 0 \\ 0 & \frac{1}{\sum_i x_i^2} \end{pmatrix} \\
\hat{\beta} &= (X'X)^{-1}X'Y \\
&= \begin{pmatrix} \frac{1}{\sum_i x_i^2} & 0 \\ 0 & \frac{1}{\sum_i x_i^2} \end{pmatrix} \begin{pmatrix} \sum_i x_i y_{i1} \\ \sum_i x_i y_{i2} \end{pmatrix} \\
&= \begin{pmatrix} \frac{\sum_i x_i y_{i1}}{\sum_i x_i^2} \\ \frac{\sum_i x_i y_{i2}}{\sum x_i^2} \end{pmatrix} \\
&= \begin{pmatrix} \hat{\beta}_1 \\ \hat{\beta}_2 \end{pmatrix} \\
RSS &= \sum_j \sum_i (y_{ji} - x_{ji}\beta_i)^2 \\
&= \sum_i (y_{i1}^2 - 2y_{i1}x_i\hat{\beta}_1 + x_{i1}^2\beta_1^2 + y_{i2}^2 - 2y_{2i}x_i + x_{2i}\beta_2^2) \\
RSS_H - RSS &= (A\beta)'[A'(X'X)^{-1}A]^{-1}(A\beta) \\
&= (\hat{\beta}_1 - \hat{\beta}_2)[\begin{pmatrix} \frac{1}{\sum_i x_i^2} & \frac{-1}{\sum_i x_i^2} \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix}]^{-1} \begin{pmatrix} \hat{\beta}_1 \\ -\hat{\beta}_2 \end{pmatrix} \\
&= (n-1)S^2 RSS_H - RSS \qquad = (A\hat{\beta})'[A'(X'X)^{-1}A]^{-1}A\hat{\beta} \\
&= \frac{n}{n+1}(\bar{Y_n} - Y_{n+1})^2 \\
&= \sum_i x_i^2 \frac{(\hat{\beta}_1 - \hat{\beta}_2)^2}{2} \\
F &= \frac{\frac{RSS_H - RSS}{1}}{\frac{(n-1)S^2}{n-1}} \\
&= \frac{\sum_i x_i^2 \frac{(\beta_1 - \beta_2)^2}{2}}{S^2} \\
&= \frac{(\beta_1 - \beta_2)^2}{2S^2(\sum_x x_i^2)^{-1}}
\end{aligned}$$

Problem 4MISC.4

$$\begin{pmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_n \\ Y_{n+1} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ \vdots & \vdots \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix} + \begin{pmatrix} \epsilon_1 \\ \epsilon_2 \\ \vdots \\ \epsilon_n \\ \epsilon_{n+1} \end{pmatrix}$$

$$H : A\hat{\beta} = 0 \implies \begin{pmatrix} 1 & -1 \end{pmatrix} \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix} = 0$$

$$X'X = \begin{pmatrix} n & 0 \\ 0 & 1 \end{pmatrix}$$

$$(X'X)^{-1} = \begin{pmatrix} 1/n & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} \hat{\mu_1} \\ \hat{\mu_2} \end{pmatrix} = \begin{pmatrix} \frac{1}{n} \sum_{i=1}^{n} Y_i \\ Y_{n+1} \end{pmatrix} = \begin{pmatrix} \bar{Y_n} \\ Y_{n+1} \end{pmatrix}$$

$$\begin{pmatrix} \hat{Y_1} \\ \hat{Y_2} \\ \vdots \\ \hat{Y_n} \\ \hat{Y_{n+1}} \end{pmatrix} = \begin{pmatrix} \bar{Y_n} \\ \bar{Y_2} \\ \vdots \\ \bar{Y_n} \\ \bar{Y_{n+1}} \end{pmatrix}$$

$$\begin{aligned} RSS &= (Y - \hat{Y})'(Y - \hat{Y}) \\ &= \begin{pmatrix} Y_1 - \bar{Y_n} & Y_2 - \bar{Y_n} & \dots Y_n - \bar{Y_n} & 0 \end{pmatrix} \begin{pmatrix} Y_1 - \bar{Y_n} \\ Y_2 - \bar{Y_n} \\ \vdots \\ Y_n - \bar{Y_n} \\ 0 \end{pmatrix} \\ &= \sum_{i=1}^{n} (Y_i - \bar{Y_n})^2 \\ &= (n-1) S_n^2 \\ F &= \frac{\frac{RSS_H - RSS}{1}}{\frac{RSS}{n+1-2}} \\ &= \frac{n}{n+1} \frac{(Y_{n+1} - \bar{Y_n})^2}{S_n^2} \end{aligned}$$