MATH-605: Homework 1

Due on Tuesday, September 12, 2017

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Contents

2.5.1

$$
\begin{aligned}
||X||_p &= (\mathbb{E}|X|^p)^{\frac{1}{p}} \\
\mathbb{E}|X|^p &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}|x|^p e^{\frac{-x^2}{2}}dx \\
&= \frac{2}{\sqrt{2\pi}}\int_{0}^{\infty}x^p e^{\frac{-x^2}{2}}dx \\
&= \frac{2}{\sqrt{2\pi}}\int_{0}^{\infty}x^{p-1} e^{\frac{-x^2}{2}}xdx \\
&= \frac{2}{\sqrt{2\pi}}\times 2^{\frac{p-1}{2}}\int_{0}^{\infty}y^{\frac{p+1}{2}-1}e^{-y}dy && [y=\frac{x^2}{2}] \\
&= \frac{2^{\frac{p+1}{2}}}{\sqrt{2\pi}}\Gamma(\frac{p+1}{2}) \\
&= 2^{\frac{p}{2}}\frac{\Gamma(\frac{p+1}{2})}{\Gamma(\frac{1}{2})} && [\because \sqrt{\pi}=\Gamma(\frac{1}{2})] \\
(\mathbb{E}|X|^p)^{\frac{1}{p}} &= 2^{\frac{1}{2}}(\frac{\Gamma(\frac{p+1}{2})}{\Gamma(\frac{1}{2})})^{\frac{1}{p}}
\end{aligned}
$$

Now using $\lim_{x\to\infty}\Gamma(x) \to x^x$

$$
\begin{aligned}
\Gamma(\frac{p+1}{2}) &\to \left(\frac{p+1}{2}\right)^{\frac{p+1}{2}} \\
\left(\Gamma(\frac{p+1}{2})\right)^{\frac{1}{p}} &\to \left(\frac{p+1}{2}\right)^{\frac{1}{2}+\frac{1}{2p}} \\
\implies (\mathbb{E}|X|^p)^{\frac{1}{p}} &= 2^{\frac{1}{2}}(\frac{\Gamma(\frac{p+1}{2})}{\Gamma(\frac{1}{2})})^{\frac{1}{p}} \to p^{\frac{1}{2}} && \text{as } p\to\infty
\end{aligned}
$$

$$
\begin{aligned}
\mathbb{E}\exp(\lambda X) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{\lambda x}e^{-\frac{x^2}{2}}dx \\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-\frac{(x-\lambda)^2}{2}}e^{\frac{\lambda^2}{2}}dx\forall\lambda\in\mathbb{R} \\
&= e^{\frac{\lambda^2}{2}} && [\because \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-\frac{(x-\lambda)^2}{2}}dx = 1]
\end{aligned}
$$

2.5.4

Assume $k = 1$ Suppose $\mathbb{E}\exp(\lambda X) \le \exp(k^2\lambda^2)$ for all $\lambda \in \mathbb{R}$ holds for $EX \ne 0$. attempting proof by contradiction:

$$\mathbb{E}\exp(\lambda X) \le \exp(\lambda^2)$$

$$E[1 + \sum_{p=1}^{\infty} \frac{(\lambda X)^p}{p!}] \le 1 + \sum_{p=1}^{\infty} \frac{\lambda^{2p}}{p!}$$

$$E[\sum_{p=1}^{\infty} \frac{(\lambda X)^p}{p!}] \le \sum_{p=1}^{\infty} \frac{\lambda^{2p}}{p!}$$

$$\sum_{p=1}^{\infty} \frac{\lambda^p}{p!}(EX^p - \lambda^p) \le 0$$

$$\lambda E[X] + \sum_{i=1} \frac{\lambda^{2i+1}E[X^{2i+1}]}{(2i+1)!} + \sum_{j=1} \lambda^{2j}(\frac{E[X^{2j}]}{2j!} - \frac{1}{j!}) \le 0$$

...... Not complete ......

Attempt 2:

Using $e^x \le x + e^{x^2} \implies \mathbb{E}[e^{\lambda X}] \le \mathbb{E}[\lambda X + e^{\lambda^2 X^2}] \le \lambda E[X] + e^{\lambda^2}$ for $\lambda \le 1$ and given $\mathbb{E}\exp(\lambda X) \le \exp(\lambda^2)$

Thus for $|\lambda| \le 1$, $\lambda E[X] \ge 0$ which can hold only if $EX = 0$. Hence $EX = 0$ is required.

2.5.5

1.
$\mathbb{E}X = 0$ and $\mathbb{E}X^2 = 1$

$$\begin{aligned}
M_{X^2}(t) &= \mathbb{E}(e^{tX^2}) \\
&= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{tx^2} e^{-\frac{x^2}{2}} dx \\
&= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}(1-2t)} dx \\
&= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2\frac{1}{(1-2t)}}} dx \\
&= \frac{1}{\sqrt{1-2t}} && \forall t > \frac{1}{2} \\
&\leq 1 && \forall t > \frac{1}{2}
\end{aligned}$$

2.

$$\begin{aligned}
\mathbb{E}(\lambda^2 X^2) &\leq \exp(K\lambda^2) \\
E[1 + \sum_{p=1} \frac{(\lambda^2 X^2)^p}{p!}] &\leq 1 + \sum_{p=1} \frac{(K\lambda^2)^p}{p!} \\
\sum_{p=1} \frac{\lambda^{2p}}{p!}(E[X^{2p}] - K^p) &\leq 0 \\
(E[X^{2p}] - K^p) &\leq 0 && \forall p \geq 1 \\
\implies E[X^{2p}] &\leq K^p && \forall p \geq 1 \\
\implies E[|X|^p]^{\frac{1}{p}} &\leq K && \forall p \geq 1 \\
\implies ||X||_\infty &< \infty
\end{aligned}$$

2.5.7

$||X||_{\psi_2} = \inf\{t > 0 : \mathbb{E}\exp(\frac{X^2}{t^2}) \leq 2\}$. A valid norml satisfies following conditions
1. $||X||_{\psi_2} \geq 0$
2. $||X||_{\psi_2} = 0$ iff $X = 0$
3. $||aX||_{\psi_2} = |a|||X||_{\psi_2}$ for $a \in \mathbb{R}$
4. $||X+Y||_{\psi_2} \leq ||X||_{\psi_2} + ||Y||_{\psi_2}$

Proof:
1. $||X||_{\psi_2} \geq 0$ as $t > 0$ always by the condition inside infimum.
2. $||X||_{\psi_2} = 0$ if $X = 0$ clearly.

For $||X||_{\psi_2} = 0 \implies X = 0$:
As $||X||_{\psi_2} = 0$, $\mathbb{E}\exp(\frac{X^2}{t^2}) \leq 2 \ \ \forall t > 0$
Assume $X \neq 0$, i.e $P(|X| > 0) > 0$ Define event $A = \{\omega \in \Omega : |X(\omega)| \geq \delta\} where \delta > 0$. Since $X \neq 0, P(A) > 0$

$$
\begin{aligned}
\exp(\frac{\delta^2}{t^2})P(A) &\leq \int_A \exp(\frac{\delta^2}{t^2})dP \\
&\leq \int_A \exp(\frac{X^2}{t^2})dP && [\because |X| > \delta on set A] \\
&\leq \mathbb{E}\exp(\frac{X^2}{t^2}) \\
&\leq 2
\end{aligned}
$$

Let $t \to 0$, then $\mathbb{E}\exp(\frac{X^2}{t^2}) > 2$ which is a contradiction, and hence $X = 0$ when $||X||_{\psi_2} = 0$
Adapted from "Subgaussian random variables: An expository note" by Omar Rivasplata.
3.

$$
\begin{aligned}
||aX||_{\psi_2} &= \inf\{t > 0 : \mathbb{E}\exp(\frac{a^2X^2}{t^2}) \leq 2\} \\
&= \inf\{|a|t > 0 : \mathbb{E}\exp(\frac{X^2}{t'^2}) \leq 2\} && [\text{Substitute } t' = |a|t] \\
&= |a| \inf\{t > 0 : \mathbb{E}\exp(\frac{X^2}{t^2}) \leq 2\} \\
&= |a|||X||_{\psi_2}
\end{aligned}
$$

4.
For 4. we make use Proposition 2.5.2 where we proved equivalence of the other forms of the norm. Here we use the p-norm form:
$||X||_{\psi_2} = \inf\{t > 0 : (\mathbb{E}|X|^p)^{\frac{1}{p}} \leq t\sqrt{p}\}$ *Lp* norm is a norm and hence satisfies triangular inequality.

$$
\begin{aligned}
||X+Y||_{\psi_2} &= \inf\{t > 0 : (\mathbb{E}|X+Y|^p)^{\frac{1}{p}} \leq t\sqrt{p}\} \\
\mathbb{E}|X+Y|^p)^{\frac{1}{p}} &\leq (E|X|^p)^{\frac{1}{p}} + (E|Y|^p)^{\frac{1}{p}} && [\text{ using Minkowski's inequality }] \\
\inf\{t > 0 : (\mathbb{E}|X+Y|^p)^{\frac{1}{p}} \leq t\sqrt{p}\} &\leq \inf\{r > 0 : (\mathbb{E}|X|^p)^{\frac{1}{p}} \leq r\sqrt{p}\} \\
&+ \inf\{s > 0 : (\mathbb{E}|Y|^p)^{\frac{1}{p}} \leq s\sqrt{p}\} \\
\implies ||X+Y||_{\psi_2} &\leq ||X||_{\psi_2} + ||Y||_{\psi_2}
\end{aligned}
$$

2.6.9

Consider X a bernoulli random variable $P(X=0)=P(X=1)=\frac{1}{2}$

$$\begin{aligned}
\mathbb{E}e^{\frac{X^2}{t^2}} &= \frac{e^{\frac{1}{4t^2}}+1}{2} \\
\mathbb{E}e^{\frac{(X-\frac{1}{2})^2}{t^2}} &= e^{\frac{1}{4t^2}} \\
||X||_{\psi_2} &= \inf\{t>0 : \mathbb{E}e^{\frac{X^2}{t^2}} \leq 2\} \\
&= \inf\{t>0 : \frac{e^{\frac{1}{4t^2}}+1}{2} \leq 2\} \\
&= \inf\{t>0 : \frac{1}{4t^2} \leq \ln(3)\} \\
&= \frac{1}{2\sqrt{\ln(3)}} \\
||X-EX||_{\psi_2} &= ||X-\frac{1}{2}||_{\psi_2} \\
||X-\frac{1}{2}||_{\psi_2} &= \inf\{t>0 : \mathbb{E}e^{\frac{(X-\frac{1}{2})^2}{t^2}} \leq 2\} \\
&= \inf\{t>0 : e^{\frac{1}{4t^2}} \leq 2\} \\
&= \frac{1}{2\sqrt{\ln(2)}}
\end{aligned}$$

Assume $||X-EX||_{\psi_2} \leq C||X||_{\psi_2}$ to be true for $C=1$, then

$$\begin{aligned}
||X-EX||_{\psi_2} &\leq ||X||_{\psi_2} \\
\implies \frac{1}{2\sqrt{\ln(2)}} &\leq \frac{1}{2\sqrt{\ln(3)}} \\
\sqrt{\ln(3)} &\leq \sqrt{\ln(2)}
\end{aligned}$$

which is a contradiction and hence $C \neq 1$

2.7.2

1. $P\{|X| \geq t\} \leq 2\exp^{-t/K_1}$ for all $t \geq 0$
2. $||X||_p = \left(E|X|^p\right)^{\frac{1}{p}} \leq K_2 p \forall p \geq 1$
3. $\mathbb{E}\exp(\lambda|X|) \leq \exp(\lambda K_3)$ for all λ such that $0 \leq \lambda\frac{1}{K_3}$
4. $\mathbb{E}\exp(|X|/K_3) \leq 2$

$1 \implies 2$:
By homogenity, X can be rescaled to X/K_1

$$
\begin{aligned}
E|X|^p &= \int_0^\infty P(|X|^p > u)du \\
&= \int_0^\infty P(|X| > t)pt^{p-1}dt && [\text{Substitute } u = t^p] \\
&\leq \int_0^\infty 2e^{-t}pt^{p-1}dt \\
&= p\Gamma(p) \\
&\leq pp^p && [\because \Gamma(p) \leq p^p] \\
(E|X|^p)^{\frac{1}{p}} &= p^{\frac{1}{p}}p \\
&\leq 2p
\end{aligned}
$$

$2 \implies 3$:

$$
\begin{aligned}
\mathbb{E}[e^{\lambda|X|}] &= \mathbb{E}[1 + \sum_{p=1}^{\infty} \frac{(\lambda|X|)^p}{p!}] \\
&= 1 + \sum_{p=1}^{\infty} \frac{\lambda^p E[|X|^p]}{p!} \\
&\leq 1 + \sum_{p=1}^{\infty} \frac{\lambda^p p^p}{p!} && [\because (E[|X|^p])^{\frac{1}{p}} \leq p] \\
&\leq 1 + \sum_{p=1}^{\infty} \lambda^p e^p && [\because p! \geq (\frac{p}{e})^p] \\
&= \frac{1}{1-\lambda e} && [\text{ for } \lambda e < 1] \\
&\leq e^{2\lambda e} && [\because \frac{1}{1-x} \leq e^2 x]
\end{aligned}
$$

$3 \implies 4$:
3 holds for $\lambda K \leq 1$ and $\exp K\lambda \to 1$ as $\lambda \to 0$
$4 \implies 1$:

$$
\begin{aligned}
E[|X|] &\leq 2 \\
P(|X| > t) &= P(e^{|X|} > e^t) \\
&= e^{-t}P(e^{|X|} > 1) \\
&\leq e^{-t}E[e^{|X|}] \\
&\leq 2e^{-t}
\end{aligned}
$$