MATH-605: Homework # 2

Due on Tuesday, September 26, 2017

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Contents

3.3.1

$X \sim \text{Unif}(\sqrt{n}S^{n-1})$
X is rotationally invariant. Consider $\mathbf{x} \in \mathbb{R}^n$ such that $||x_i||_2 = ||x||_2 \forall i \in [1, n]$

$$\begin{aligned}\mathbb{E}[\sum_{i=1}^{n} \langle X, x_i \rangle^2] &= n\mathbb{E}[\langle X, x \rangle^2] \\ &= \mathbb{E}[||X||_2^2 ||x||_2^2] \\ &= n||x||_2^2 \\ \implies \mathbb{E}[\langle X, x \rangle^2] &= ||x||_2^2\end{aligned}$$

and hence X is isotropic

3.3.3

Consider $X_i \sim N(0, \sigma^2)$

$$\begin{aligned}M_{X_i}(t) &= \mathbb{E}[e^{tX_i}] \\ &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{tx_i} e^{-\frac{x_i^2}{2\sigma^2}} \\ &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma_i^2}} e^{\frac{t^2\sigma^2}{2}} e^{-(\frac{x_i}{\sigma} - t\sigma)^2} & = e^{\frac{t^2\sigma^2}{2}}\end{aligned}$$

We use uniqueness property of MGF (MGF $\leftrightarrow$ distribution) in the parts that follow.
Part 1.
$g \sim \mathcal{N}(0, I_n)$

$$\begin{aligned}\langle g, u \rangle &= u^T g \\ M_{\langle g, u \rangle}(t) &= e^{\frac{t^2 u^T u}{2}} \\ \implies \langle g, u \rangle &= N(0, u^T u) \\ &= N(0, ||u||_2^2)\end{aligned}$$

Part 2.

$$\begin{aligned}M_{X_i}(t) &= e^{\frac{t^2\sigma_i^2}{2}} \\ M_{\sum X_i}(t) &= e^{\frac{t^2 \sum \sigma_i^2}{2}} \\ \implies \sum X_i &= N(0, \sum_i \sigma_i^2)\end{aligned}$$

Part 3.
$G_{ij} \sim N(0, 1)$. From Part 1 $G_{ij}u_i \sim N(0, u_i^2) = N(0, 1) \implies Gu \sim N(0, I_m)$

3.3.5

$X \sim N(0, I_n)$
Part 1.

$$\begin{aligned}
\mathbb{E}[\langle X,u\rangle\langle X,v\rangle] &= \mathbb{E}[u^TX(X^Tv)^T] \\
&= \mathbb{E}[v^TXX^Tu] && \text{[Since } v^TX \text{ and } u^TX \text{ are both scalars]} \\
&= v^T\mathbb{E}[XX^T]u \\
&= v^Tu \\
&= \langle u,v\rangle
\end{aligned}$$

Part 2.

$$\begin{aligned}
||Xu - Xv||_{L^2}^2 &= \mathbb{E}[(u^TX - v^TX)(u^TX - v^TX)^T] \\
&= \mathbb{E}[u^TXX^Tu + v^TXX^Tv - u^TXX^Tv - v^TXX^Tu] \\
&= u^T\mathbb{E}[XX^T]u + v^T\mathbb{E}[XX^T]v - u^T\mathbb{E}[XX^T]v - v^T\mathbb{E}[XX^T]u \\
&= (u-v)(u-v)^T \\
\implies ||Xu - Xv||_{L^2} &= ||u-v||_{L^2}
\end{aligned}$$

3.3.6

From 3.3.5. $\mathbb{E}[\langle G,u\rangle\langle G,v\rangle] = \langle u,v\rangle$ Since u, v are orthogonal $\langle u,v\rangle = 0$ Hence $\mathbb{E}[\langle G,u\rangle\langle G,v\rangle] = 0$.
Also, $Gu \sim N(0, ||u||_2^)$ and $Gv \sim N(0, ||v||_2^)$ and from above we get $\mathbb{E}[\langle G,u\rangle\langle G,v\rangle] = 0 = \mathbb{E}[\langle G,u\rangle] = \mathbb{E}[\langle G,v\rangle]$. Since Gu and Gv are both gaussian, and $GuGv$ have zero correlation, it implies independence.

3.5.3

I tried a few configurations of $A = A^T$, but couldn't disprove the inequality. E.g. $\frac{1}{3}\begin{pmatrix} \frac{1}{2} & 1 \\ 1 & \frac{1}{2} \end{pmatrix}$ then $\sum |A_{ij}x_ix_j| = \frac{1}{2}(x_1 + x_2)^2 + 2x_1x_2 \leq \frac{3}{3} = 1$, however for $u_1 = v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$; $u_2 = v_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ $\sum_{A_{ij}u_iu_j} = 1$

3.6.7

Consider a 2 dimensional version of the problem where the n dimensional vector g is projected to the plane which contains u and v. Since g is normally distributed, the hyperplane g is now a line and is distributed uniformly over a unit circle. See Figure 1.
For $\langle g, u\rangle$ and $\langle g, v\rangle$ to have opposite signs, the line g passes through between u and v and the probability of this is given by $\frac{\alpha}{\pi}$ where $cos(\alpha) = \langle u, v\rangle$
Thus,

$$\begin{aligned}
\mathbb{E}[sign(g,u)sign(g,v)] &= -1 \times P(sign(g,u)sign(g,v) = -1) \\
&+ 1 \times P(sign(g,u)sign(g,v) = 1) \\
&= -1(\frac{\arccos\langle u,v\rangle}{\pi}) + 1(1 - \frac{\arccos\langle u,v\rangle}{\pi}) \\
&= 1 - 2\frac{\arccos\langle u,v\rangle}{\pi} \\
&= 1 - 2(\frac{\pi}{2\pi} - \frac{\arcsin\langle u,v\rangle}{\pi}) && [\because \arcsin\theta + \arccos\theta = \frac{\pi}{2}] \\
&= 2\frac{\arcsin\langle u,v\rangle}{\pi}
\end{aligned}$$

3.7.5

Part 1.
Consider $\phi(u) = \sqrt{2}u \otimes u \oplus \sqrt{5}u \otimes u \otimes u$ and $\phi(v) = \sqrt{2}v \otimes v \oplus \sqrt{5}v \otimes v \otimes v$ and u, v are orthogonal.

$$\begin{aligned}
\langle\phi(u), \phi(v)\rangle &= \langle\sqrt{2}u \otimes u \oplus \sqrt{5}u \otimes u \otimes u, \sqrt{2}v \otimes v \oplus \sqrt{5}v \otimes v \otimes v\rangle \\
&= \langle\sqrt{2}u \otimes u, \sqrt{2}v \otimes v\rangle \oplus \langle\sqrt{5}u \otimes u \otimes u, \sqrt{5}v \otimes v \otimes v\rangle + 0 \\
&= 2\langle u \otimes u, v \otimes v\rangle + 5\langle u \times u \otimes u, v \otimes v \otimes v\rangle \\
&= 2\langle u, v\rangle^2 + 5\langle u, v\rangle^3
\end{aligned}$$

Part 2.
For $f(\langle u, v\rangle) = \sum_{i=0}^{k} a_i\langle u, v\rangle^i$, we let $\phi(u) = \sum_{i=0}^{k} a_i u^{\otimes i}$ and $\phi(v) = \sum_{i=0}^{k} v^{\otimes i}$
Part 3.
$f(x) = \sum_{k=0}^{\infty} a_k x^k$, since $a_k > 0$, $\phi(u) = \sum_{k=0}^{\infty} \sqrt{a_k}u^{\otimes k}$ and $\phi(v) = \sum_{k=0}^{\infty} \sqrt{a_k}v^{\otimes k}$

3.7.6

Let $\phi(u) = \sum_{k=0} \sqrt{a_k}u^{\otimes k}$ and $\psi(v) = sign(a_k)\sqrt{a_k}v^{\otimes k}$ Then $\langle\phi(u), \psi(v)\rangle = \langle\sum_{k=0} \sqrt{a_k}u^{\otimes k}, \sum_{k=0} sign(a_k)\sqrt{a_k}u^{\otimes k}\rangle = \sum_k a_k\langle u, v\rangle^k$
Hence $||\phi(u)|| = ||\psi(u)|| = \langle\sum_{k=0} \sqrt{a_k}u^{\otimes k}, \sum_{k=0} sign(a_k)\sqrt{a_k}v^{\otimes k}\rangle = |\sum_{k=0} a_k\langle u, u\rangle^k|$

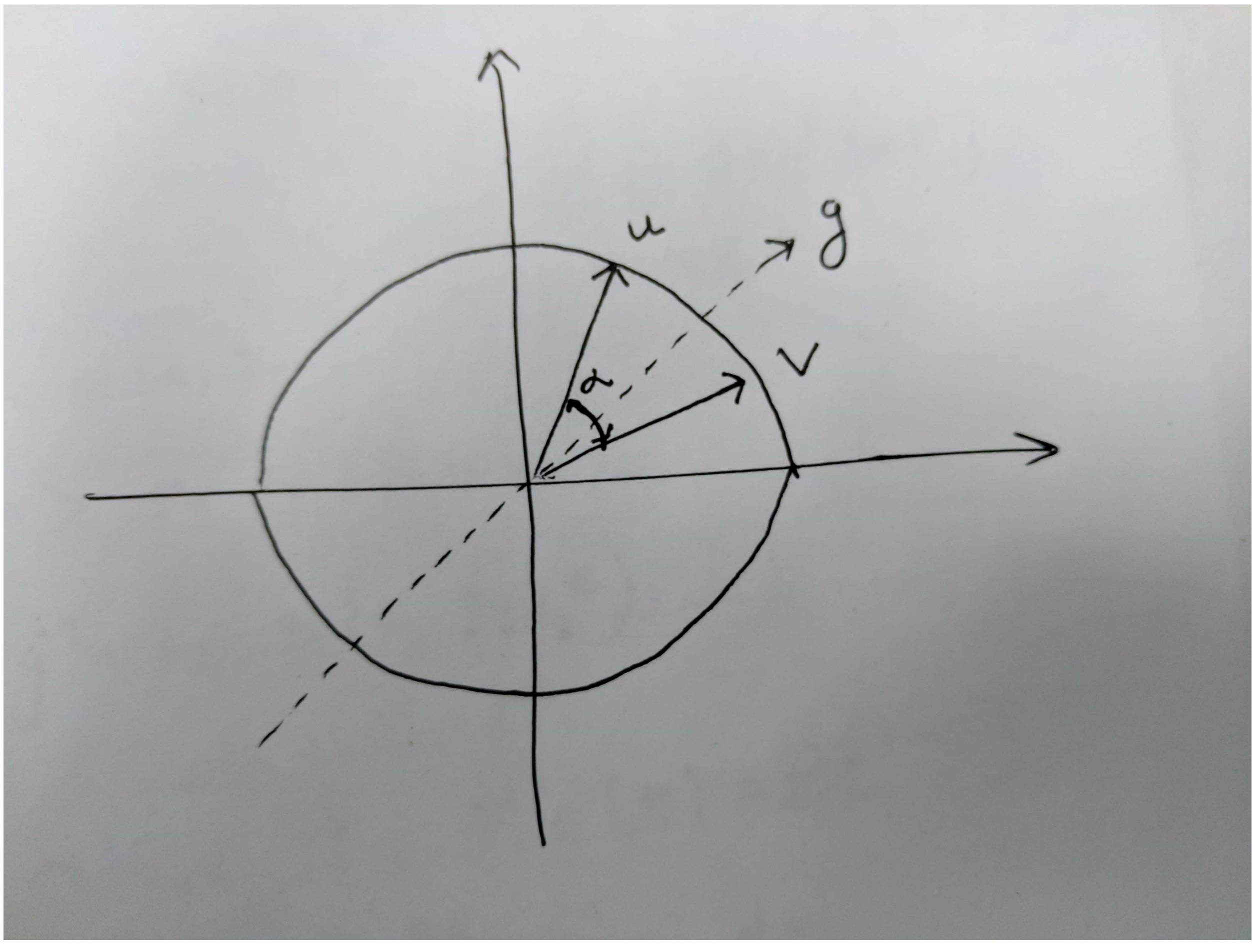

Figure 1: Problem 3.6.7 unit circle. g causes $\langle g, u\rangle$ and $\langle g, v\rangle$ to have opposite sings if it lies in the sector encompassed by u and v