EE-546: Assignment # 2

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Contents

Problem # 1

Problem 1a):

$$
\begin{aligned}
Z &= \frac{1}{m}\sum_{r=1}^{m} X_r|X_r| \text{ sign}(X_r + \mathbf{a_r}^T\mathbf{y}) \\
&= \frac{1}{m}\sum_{r=1}^{m} X_r^2 \text{ sign}(X_r)\text{sign}(X_r + \mathbf{a_r}^T\mathbf{y}) \\
\text{Let } Z_r &= X_r^2 \text{ sign}(X_r)\text{sign}(X_r + \mathbf{a_r}^T\mathbf{y}) \\
\mathbf{a_r}^T\mathbf{y} &\sim \mathcal{N}(0, \mathbf{y^T y}) \\
\implies \frac{\mathbf{a_r}^T\mathbf{y}}{\sqrt{\mathbf{y^T y}}} &\sim \mathcal{N}(0,1) \\
\mathbb{E}[Z_r] &= \mathbb{E}[X_r^2\text{sign}(X_r)\text{sign}(X_r + \mathbf{a_r}^T\mathbf{y})] \\
&= \mathbb{E}[X_r^2\text{sign}(X_r)\text{sign}(X_r + \sqrt{\mathbf{y^T y}}\frac{\mathbf{a_r}^T\mathbf{y}}{\sqrt{\mathbf{y^T y}}})] \\
&= \frac{2}{\pi}\tan^{-1}\frac{1}{\sqrt{\mathbf{y^T y}}} + \frac{2}{\pi}\frac{\sqrt{\mathbf{y^T y}}}{1+\mathbf{y^T y}} \\
\implies \mathbb{E}[Z] &= \frac{1}{m}\sum_{r=1}^{m}\mathbb{E}[Z_r] \\
&= \frac{2}{\pi}\tan^{-1}\frac{1}{\sqrt{\mathbf{y^T y}}} + \frac{2}{\pi}\frac{\sqrt{\mathbf{y^T y}}}{1+\mathbf{y^T y}}
\end{aligned}
$$

Problem 1b):

$$
\begin{aligned}
\mathbb{P}(|X_r^2 \text{ sign}(X_r)\text{sign}(X_r + \mathbf{a_r}^T\mathbf{y})| \geq t) &= \mathbb{P}(|X_r^2| \geq t) \qquad \because \text{sign is immaterial} \\
&= P(\chi_1^2 \geq t)
\end{aligned}
$$

$P(\chi^2 \geq t)$ is easily obtained as χ^2 is sub-exponential,
$Z \sim \chi_m^2$ and hence:

$$P(|Z - \mathbb{E}[Z]| \geq t) \leq \exp(1 - \frac{t}{k_1})$$

Problem # 2

Problem 2 (i):
From Cauchy Schwartz inequality we have:

$$|\langle x, y\rangle| \leq ||x||||y||$$

Thus,

$$\begin{aligned}
\langle x, Ay\rangle &\leq ||x||\ ||Ay|| \\
\max_{||x||_{l_2}=1, ||y||_{l_2}=1} \langle x, Ay\rangle &\leq \max_{||x||_{l_2}=1, ||y||_{l_2}=1} ||x||\ ||Ay|| \\
&= \max_{||y||_{l_2}=1} ||Ay|| && \because ||x|| = 1 \\
&= ||A|| && \because \text{from the definition of } ||A||
\end{aligned}$$

Fact: For unitary U, $U^TU = I \implies ||Ux||_{l_2}^2 = x^TU^TUx = ||x||_{l_2}^2$
Now consider $||Ax||$:

$$\begin{aligned}
\sup_{||x||=1} ||Ax|| &= \sup_{||x||=1} ||U\Sigma V^Tx|| && \because \text{ spectral decomosition of A} \\
&= \sup_{||x||=1} ||\Sigma V^Tx|| && \because ||Ux|| = ||x|| forunitaryU \\
&= \sup_{||y||=1} ||\Sigma y|| && \because y = V^Tx \text{ and } y^Ty = 1
\end{aligned}$$

Since Σ is a diagnoal matrix $\sup_{||y||=1} ||\Sigma y||$ can easily be obtained when $y = (1, 0, \dots, 0)$ such that the maximum will be $\sigma_1(A)$
Problem 2 (ii):

$$\begin{aligned}
\text{trace}(U^TAV) &= \sum_{i=1}^{r} u_i^T a_{ii} v_i \\
\max \text{trace}(U^TAV) &= \max \sum_{i=1}^{r} u_i^T a_{ii} v_i \\
&= \sum_{i=1}^{r} \sigma_i(A)
\end{aligned}$$

Problem # 3

Upper bound:

$$
\begin{aligned}
||A|| = \sup_{||x||=1} ||Ax|| &= \sup_{||x||=1} \sqrt{\sum_{i=1}^{m} \big(\sum_{j=1}^{n} A_{ij}x_j\big)^2} \\
&\leq \sup_{||x||=1} \sqrt{\sum_{i=1}^{m}(\sum_{j=1}^{n} A_{ij}^2)(\sum_{k=1}^{n} x_k^2)} \qquad \because \text{ using Cauchy-Schwartz inequality} \\
\implies ||A|| &\leq \sqrt{m} \max_{i\in\{1,2,\dots,m\}} \big(\sum_{j=1}^{n} A_{ij}^2\big)^{1/2}
\end{aligned}
$$

For eqaulity $A_{ij}x_j = \lambda x_j$ and hence $A = I$ is one such matrix.
Lower bound:
Consider $x = (\frac{1}{\sqrt{n}}, \frac{1}{\sqrt{n}} \dots, \frac{1}{\sqrt{n}})$
them

$$
\begin{aligned}
||A|| = \sup_{||x||=1} ||Ax|| &\geq \sqrt{\sum_{i=1}^{m} \big(\sum_{j=1}^{n} A_{ij}\frac{1}{\sqrt{n}}\big)^2} \\
&= \frac{1}{\sqrt{n}} \sqrt{\sum_{i=1}^{m} \big(\sum_{j=1}^{n} A_{ij} \times 1\big)^2} \\
&\geq \frac{1}{\sqrt{n}} \sqrt{\frac{1}{m}\big(\sum_{i=1}^{m} |\sum_{j=1}^{n} A_{ij}|\big)^2} \\
&= \frac{1}{\sqrt{mn}} \sum_{i=1}^{m} |\sum_{j=1}^{n} A_{ij}|
\end{aligned}
$$

Problem # 4

$$
\begin{aligned}
||A||_F &= (\sum_{i=1}^{m}\sum_{j=1}^{n}|a_{ij}|^2)^{\frac{1}{2}} \\
&= (\mathrm{Tr}(A^TA))^{\frac{1}{2}} \\
&= (\mathrm{Tr}(V\Sigma U^T U\Sigma V^T))^{\frac{1}{2}} && \because \text{Spectral decomposition of } A \\
&= (\mathrm{Tr}(V\Sigma^2 V^T))^{\frac{1}{2}} \\
&= (\mathrm{Tr}(V^T V\Sigma^2))^{\frac{1}{2}} && \because \mathrm{Tr}(AB) = \mathrm{Tr}(BA) \\
&= (\mathrm{Tr}(\Sigma^2))^{\frac{1}{2}} \\
&= (\sum_{i=1}^{\min(m,n)} \sigma_i^2(A))^{\frac{1}{2}}
\end{aligned}
$$

From Problem 2 $||A|| = \sigma_1(A)$. Now, given $\sigma_1(A) \geq \sigma_2(A) \geq \cdots \geq \sigma_{\min(m,n)}(A)$

$$
\begin{aligned}
\sigma_1(A) &\leq \sqrt{\sigma_1(A)^2 + \sigma_2(A)^2 + \cdots + \sigma_{\min(m,n)}(A)^2} \\
\sqrt{\sigma_1(A)^2 + \sigma_2(A)^2 + \cdots + \sigma_{\min(m,n)}(A)^2} &\leq \sqrt{\sigma_1(A)^2 + \sigma_1(A)^2 + \cdots + \sigma_1(A)^2} \\
&= \sqrt{\mathrm{rank}(A)}\sigma_1(A)
\end{aligned}
$$

where the l;ast equality follows from the fact that only rank(A) singular vectors are non zero.
Thus,

$$
\sigma_1(A) \leq \sqrt{\sigma_1(A)^2 + \sigma_2(A)^2 + \cdots + \sigma_{\min(m,n)}(A)^2} \leq \sqrt{\mathrm{rank}(A)}\sigma_1(A)
$$

$\implies$

$$
\sigma_1(A) \leq ||A|| \leq \sqrt{\mathrm{rank}(A)}||A||
$$