EE599 Homework 1

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1 Problem 1

$x(k+1) = x(k) + 1 \; \forall k \geq 1$

$$\begin{aligned} x(1) &= 1 \\ x(2) &= 2 \\ &\vdots \\ x(n) &= n \end{aligned}$$

Also, $\mathbf{x}_0 = [-1, 1]^T$ $\mathbf{x}_k = [x(k), x(k+1)]^T \; \forall k \geq 1$

1.1 Problem 1a

If $\mathbf{x}_{k_1}, \mathbf{x}_{k_2}$ form a basis of $\mathbb{R}^2$, they must be linearly independent, i.e. $\alpha_1 \mathbf{x}_{k_1} + \alpha_2 \mathbf{x}_{k_2} = 0$ should hold iff $\alpha_1 = \alpha_2 = 0$.

Case 1: When $k_1 \neq k_2 \geq 1$

$$\alpha_1 \mathbf{x}_{k_1} + \alpha_2 \mathbf{x}_{k_2} = 0$$
$$\begin{pmatrix} \alpha_1 k_1 + \alpha_2 k_2 \\ \alpha_1(k_1+1) + \alpha_2(k_2+1) \end{pmatrix} = 0 \; \forall k_1 \neq k_2 \geq 1$$

$\implies \alpha_1 k_1 + \alpha_2 k_2 = 0$ and $\alpha_1 k_1 + \alpha_2 k_2 + \alpha_1 + \alpha_2 = 0$

$\implies \alpha_1 + \alpha_2 = 0 \; \forall k_1 \neq k_2 \geq 1$

$\implies \alpha_1 = \alpha_2 = 0$ OR $\alpha_1 = -\alpha_2$ But $\alpha_1 = \alpha_2 \implies k_1 = k_2$ as $\alpha_1 k_1 + \alpha_2 k_2 = 0$ which is not possible since $k_1 \neq k_2$. Thus $\alpha_1 = \alpha_2 = 0$

Case 1: When $k_1 = 0, k_2 \geq 1$

For the case when one the vectors is $\mathbf{x}_0$:

$$\alpha_1 \mathbf{x}_0 + \alpha_2 \mathbf{x}_{k_2} = 0$$
$$\begin{pmatrix} -\alpha_1 + \alpha_2 k_2 \\ \alpha_1 + \alpha_2(k_2+1) \end{pmatrix} = 0 \; \forall k_2 \geq 1$$

$\implies \alpha_1 = \alpha_2 k_2$ and $\alpha_1 = -\alpha_2(k_2+1)$ which is possible only if $\alpha_1 = \alpha_2 = 0$

1.2 Problem 1b

Let ϕ_0, ϕ_1 to be orthonormal bases.
$\mathbf{x}_k = [k, k+1]^T$
$||\mathbf{x}_0|| = \frac{1}{\sqrt{2}}[-1, 1]^T$

$$\begin{aligned}
\phi_0 &= \frac{x_0}{||x_0||} \\
&= \frac{1}{\sqrt{2}}[-1, 1]^T \\
v_1 &= \langle x_1, \phi_0 \rangle \phi_0 \\
&= \frac{1}{2}[-1, 1]^T \\
\phi_1 &= \frac{x_k - v_1}{||x_k - v_1||} \\
&= \frac{1}{\sqrt{2}(k + \frac{1}{2})}[k + \frac{1}{2}, k + \frac{1}{2}]^T
\end{aligned}$$

Thus, $\phi_0 = \frac{1}{\sqrt{2}}[-1, 1]^T$ and $\phi_1 = \frac{1}{\sqrt{2}(k+\frac{1}{2})}[k + \frac{1}{2}, k + \frac{1}{2}]^T$

2 Problem 2

2.1 Problem 2a

Since T_2 orthogonal transform induces rotation, and vector v_2 lies on the unit circle, the upper bound of l_∞ norm of $T_2\dot{v}$ will be 1 itself, since rotation cannot increase the maximum norm. The lower bound achieved when both components of $\mathbf{v} = [v_1, v_2]$ are equal, i.e. $v_1 = v_2 = \frac{1}{2}$

Thus, $a_2 = \frac{1}{2}$ and $b_2 = 1$

2.2 Problem 2b

For the $n > 2$ case, the maximum norm will still remain one as the vector now lies on the surface of an n-dimensional sphere. And hence $a_n = 1$. the lower bound is achieved vis symmetry when all components of vector $\mathbf{v}_n = [v_1, v_2, \ldots v_n]$ are equal and $v_1 = v_2 \cdots = v_n = \frac{1}{n}$ and hence $a_n = \frac{1}{\sqrt{n}}$ and $b_n = 1$

3 Problem 3

$\alpha_i = \langle x_i, y \rangle$ where x_i form an orthonormal basis.

$$
\begin{aligned}
y &= \sum_{i=1}^{n} \alpha_i x_i \\
\hat{y} &= \sum_{i=i}^{k} \beta_i x_i \\
||y - \hat{y}|| &= || \sum_{i=1}^{n} \alpha_i x_i - \sum_{i=1}^{k} \beta_i x_i || \\
&= || \sum_{i=1}^{k} (\alpha_i - \beta_i) x_i + \sum_{i=k+1}^{n} \alpha_i x_i ||
\end{aligned}
$$

Using Parseval's equlity with orthonormal basis:

$$
\begin{aligned}
||y - \hat{y}||^2 &= \sum_{i=1}^{n} |\langle x_i, y - \hat{y} \rangle|^2 \\
\langle x_i, y - \hat{y} \rangle &= \langle x_i, \sum_{i=1}^{k} (\alpha_i - \beta_i) x_i + \sum_{i=k+1}^{n} \alpha_i x_i \rangle \\
&= \begin{cases} \alpha_i - \beta_i & 1 \le i \le k \\ \alpha_i & k+1 \le i \le n \end{cases} \\
||y - \hat{y}||^2 &= \sum_{i=1}^{m} (\alpha_i - \beta_i)^2 + \sum_{i=k+1}^{n} \alpha_i^2
\end{aligned}
$$

Thus in order to minimize $||y - \hat{y}||^2$, we need to minimize $\sum_{i=1}^{m} (\alpha_i - \beta_i)^2 + \sum_{i=k+1}^{n} \alpha_i^2$, which will require $\alpha_i = \beta_i \; \forall \; 1 \le i \le k$

4 Problem 4

4.1 Problem 4a

$f_s(x), f_a(x)$ can be written as:

$$
\begin{aligned}
f_s(x) &= \frac{1}{2}(f(x) + f(-x)) \\
f_a(x) &= \frac{1}{2}(f(x) - f(-x)) \\
\implies f_(x) &= f_s(x) + f_a(x)
\end{aligned}
$$

4.2 Problem 4b

On S, $f(x) = f(-x)$ and hence an orthonormal basis is given by the symmetric basis $\{\frac{1}{\sqrt{2\pi}}, \frac{\cos nx}{\sqrt{\pi}}\}$ while for A, $f(x) = -f(-x)$ and hence a choice of basis would be $\{\frac{\sin nx}{\sqrt{\pi}}\}$

4.3 Problem 4c

For $f(x) \in L_2[-\pi,\pi]$ we have $f(x) = f_s(x) + f_a(x)$ thus, $L_2[-\pi,\pi] \subseteq S \oplus A$

Given orthonormal bases of S, A and $f_s(x) + f_a(x) = f(x)$, so the sum of functions $f_s(x) \in S$ and $f_a(x) \in A$ can be equated to some $f(x) \in L_2[-\pi,\pi]$ and hence $S \oplus A \subseteq L_2[-\pi,\pi]$

Thus, $S \oplus A = L_2[-\pi,\pi]$.

5 Problem 5

5.1 FFT

The FFT shows clear peaks at 20Hz and 80Hz indicating the signal is ea mixture of signals of these frequencies.

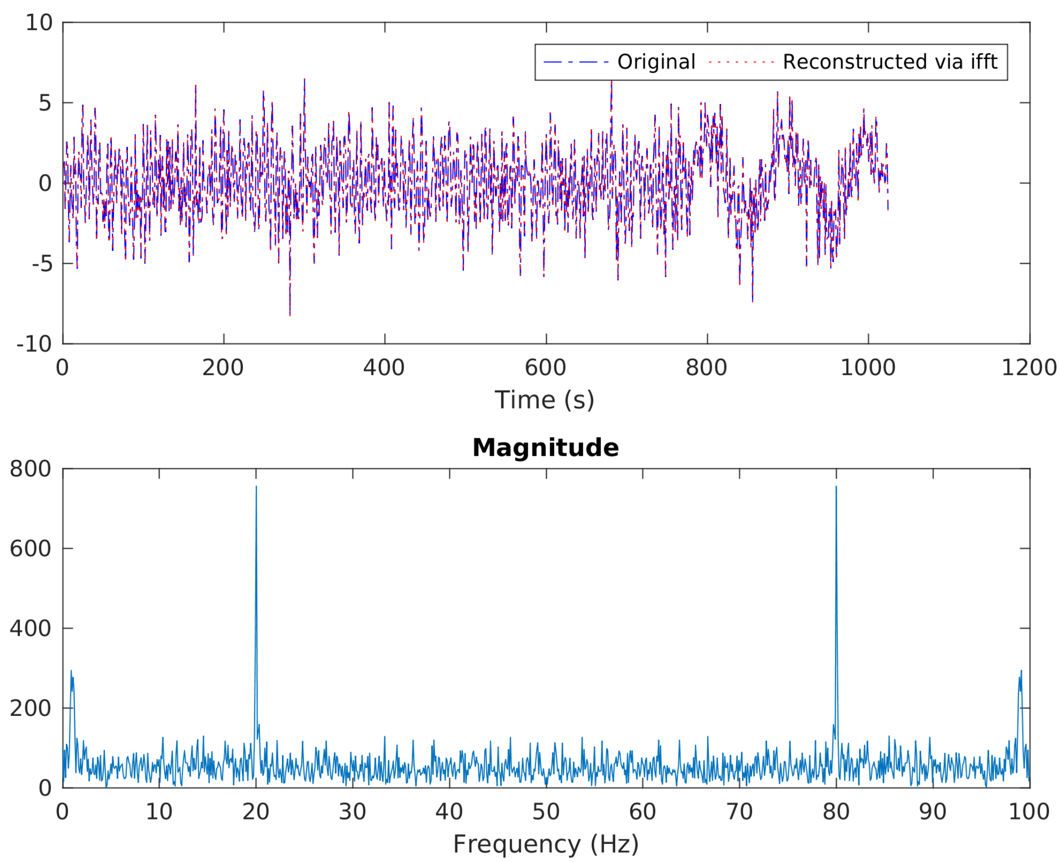


Figure 1: Problem 5: Original signal, reconstructed signal and Fourier transform

Check that the set of vectors generated by the GenerateBases.m is a basis. 1. (1 pt.) Check that the set of vectors generated by the GenerateBases.m is a basis. 2. (1 pt.)Is the basis orthogonal? Provide matlab code to verify your answer. 3. (2 pt.)Compute the projection of the input signal onto each of the basis vector. 4. (6 pts.) Generate a time-frequency plot using the projections. You can simply plot the square of each projection on the time-frequency tile of the corresponding basis vector. An example of how to do this is provided in the m-file HW1 MATLAB.m. 5. (6 pts.) What does this time-frequency plot tell you about the signal? How is this interpretation of the signal different than when using an FFT? How is this interpretation of the signal different than when using the other basis?

5.2 Checking if generated vectors form a basis

Check for rank of each bases, it should be equal to $N = 1024$

`assert(rank(T_dct)==N)` and `assert(rank(T_haar)==N)`

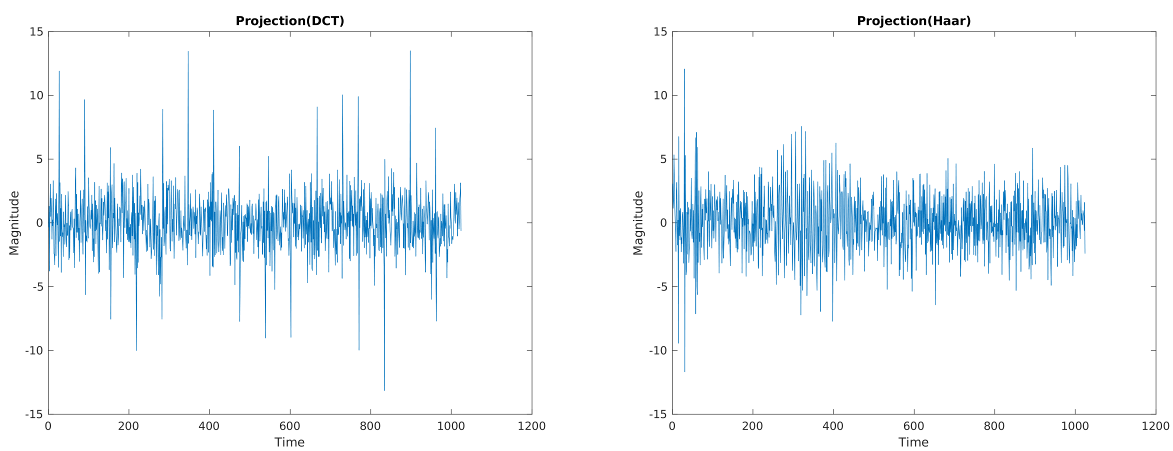
5.3 Checking Orthogonality

$\langle \phi_i, \phi_k \rangle = \delta_{i-k}$ where $\delta_{i=k} = 1$ if $i = k$ and 0 otherwise. So check if matrices $T_dct{*}T_dct$ and $T_haar{*}T_haar$ are identity matrices $\mathbf{I}_N$

```
epsilon = 1e-6;
T_dct_prod = T_dct * transpose(T_dct);
T_haar_prod = T_haar * transpose(T_haar);

assert(sum(sum(T_dct_prod - eye(size(T_dct_prod,1)) < epsilon)) == 0);
assert(sum(sum(T_haar_prod - eye(size(T_haar_prod,1)) < epsilon)) == 0);
```

5.4 Projection



(a) DCT

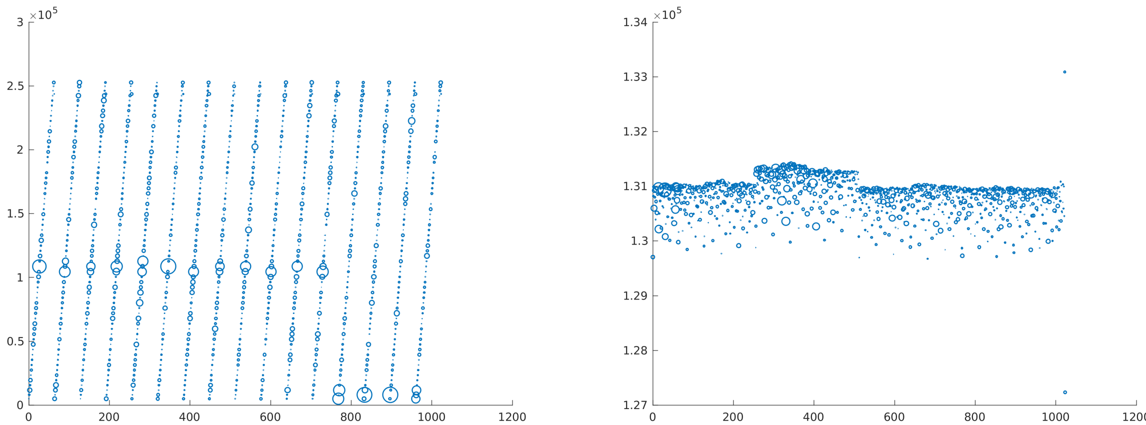
(b) Haar

Figure 2: Projected signal of DCT($N_b = 64$) and Haar

```
c_dct = zeros(1, N);
c_haar = zeros(1, N);
i=1;
```

```
while i<=N
    c_dct(1, i) = dot(T_dct(i,:), x.');
    c_haar(1, i) = dot(T_haar(i,:), x.');
    i = i+1;
end
```

5.5 T-F plot of projections



(a) DCT

(b) Haar

Figure 3: T-F plot of DCT($N_b = 64$) and Haar

5.6 Interpretation

DCT's T-F plot shows a uniform tiling where the power is uniformly distributed across time and scale. For the Haar , it appears to be concentrated across a limited band and is not uniformly distributed across. The FFT plot only shows frequency distribution and is not localized in time, while these plots are both localized in time and frequency.

5.7 New basis

For the new basis I use a smaller block size $N_b = 4$.

5.7.1 Checking if generated vectors form a basis

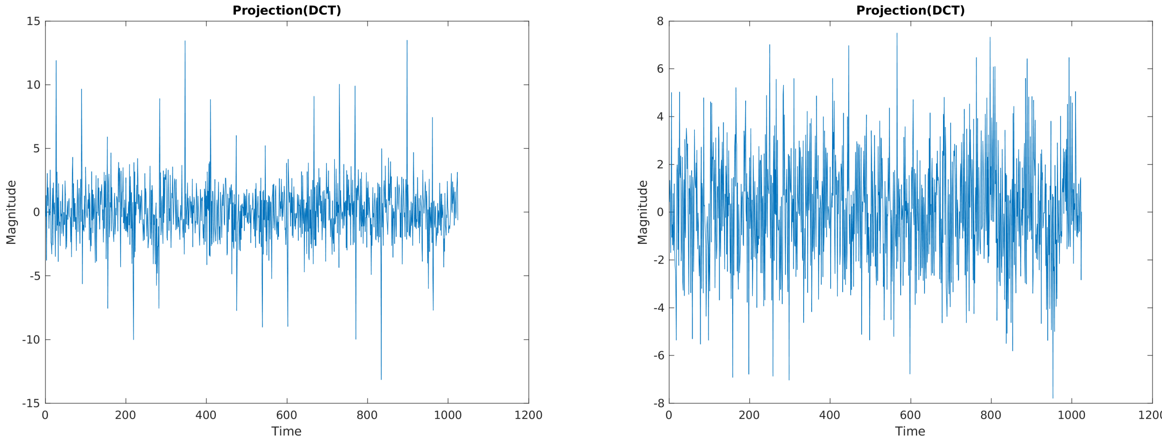
Check for rank of each bases, it should be equal to $N = 1024$

```
assert(rank(T_dct)==N)
```

5.7.2 Checking Orthogonality

$\langle \phi_i, \phi_k \rangle = \delta_{i-k}$ where $\delta_{i=k} = 1$ if $i = k$ and 0 otherwise. So check if matrices $T_dct * T_dct$ are identity matrices $\mathbf{I}_N$

5.7.3 Projection

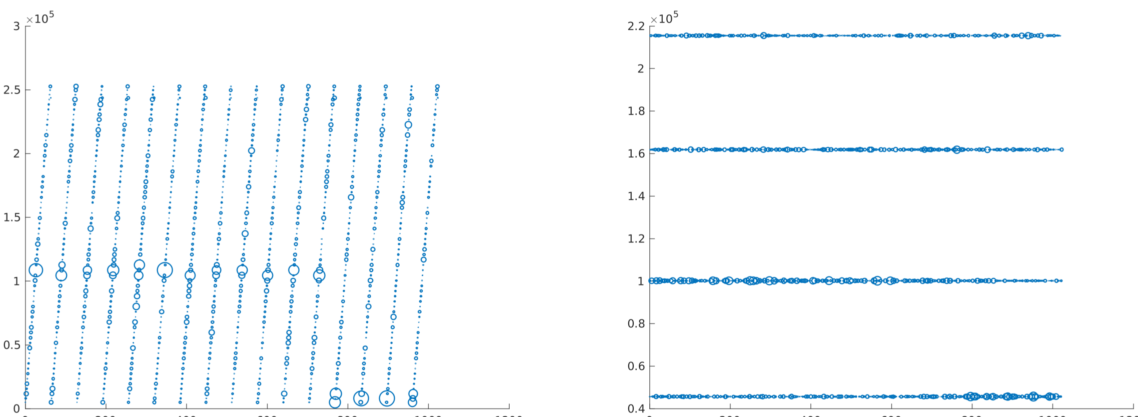


(a) DCT($N_b = 64$)

(b) DCT($N_b = 4$)

Figure 4: Projected signal of DCT($N_b = 64$) and DCT($N_b = 4$)

5.7.4 T-F plot of projections



(a) DCT($N_b = 64$)

(b) DCT($N_b = 4$)

Figure 5: T-F plot of DCT($N_b = 64$) and and DCT($N_b = 4$)