Problem 1 (35 pts). (Note: All questions can be solved independently)

Define the following time series (for $k \geq 1$):

$$x(1) = 1 \quad \text{and} \quad x(k+1) = x(k) + 1, \quad k > 1.$$

Use this to define a set, $\mathcal{S}$, of vectors in R^2 indexed by a non-negative integer k:

$$\mathcal{S} = \{\mathbf{x}_k\}_{k \in Z, k \geq 1}.$$

where

$$\mathbf{x}_k = [x(k) \;\; x(k+1)]^t, \quad k \geq 1, \quad \text{and}$$
$$\mathbf{x}_0 = [-1, 1],$$

with ^{t} denoting the transpose.

a. (5 pts) Plot in the 2D plane $\mathbf{x}_0$, $\mathbf{x}_1$, $\mathbf{x}_2$, and $\mathbf{x}_3$.

b. (10 pts) Prove that for all $k_1, k_2 \geq 0$, $k_1 \neq k_2$ we have that $\mathbf{x}_{k_1}, \mathbf{x}_{k_2}$ form a basis for R^2.

c. (10pts) Design an orthogonal basis for R^2, taking as a starting point a pair of vectors $\mathbf{x}_0$, $\mathbf{x}_k$, with $k \neq 0$.

d. (10pts) We wish to create a dictionary $\mathcal{S}_D \subset \mathcal{S}$ by selecting vectors in $\mathcal{S}$. Our goal in designing such a dictionary is minimizing the maximum coherence, i.e., minimizing

$$\max_{k_1,k_2 \in \mathcal{S}_D} \frac{1}{||x_{k_1}|| \ ||x_{k_2}||} < \mathbf{x}_{k_1}, \mathbf{x}_{k_2} >$$

where $||\mathbf{x}_k||$ is the norm of $\mathbf{x}_k$ and $< ., . >$ denotes the inner product. Prove that the minimum coherence dictionary of size d will have the form:

$$\mathcal{S}_D = \{\mathbf{x}_0\} \cup \{\mathbf{x}_{k_1}, \mathbf{x}_{k_2}, \ldots \mathbf{x}_{k_{d-1}}\},$$

that is, $\mathbf{x}_0$ should always be part of such a minimum coherence dictionary.