Problem 2 (30 pts). (Note: All questions can be solved independently)

Let $x(n)$ be a sequence in $l_2(\mathbb{Z})$, with $X(e^{j\omega})$ its corresponding discrete time Fourier transforms. For each of the sets below either prove that they form subspaces in $l_2(\mathbb{Z})$ or provide a counter-example to show that they are not subspaces.

a. (6 pts) $S_{\omega_1} = \{x(n) \in l_2(\mathbb{Z}) \text{ such that } |X(e^{j\omega})| = 0 \;\; \forall\omega \in [-2\pi, -\omega_1] \cup [\omega_1, 2\pi]\}$ for a given $\omega_1 \in (0, 2\pi)$.

b. (7 pts) $S_{\omega_1} \cap S_{\omega_2}$ with $\omega_1 \neq \omega_2$

c. (7 pts) $S_{\omega_1}^{\Delta} = \{x(n) \in S_{\omega_1}$ such that $||X(e^{j\omega})||^2 \leq \Delta\}$ for a given $\Delta > 0$.

d. (10 pts) Find the orthogonal complement of each of the sets above (questions (a), (b), (c)).

Problem 3 (25 pts). (Note: All questions can be solved independently)

Let $\mathbf{A}$ and $\mathbf{B}$ be two different real-valued $N \times N$ orthogonal matrices. Let $\mathbf{x} \in \mathbb{R}^N$ be an arbitrary vector and let $\mathbf{y}_a = \mathbf{A}\mathbf{x}$ and $\mathbf{y}_b = \mathbf{B}\mathbf{x}$. Define a new matrix $\mathbf{T}$ with $2N$ columns of size N, containing all the columns of $\mathbf{A}$ and $\mathbf{B}$. Denote $|| \cdot ||_2$ the l_2 norm of a vector.

a. (7 pts) Compute $||\mathbf{y}_a||_2^2 - ||\mathbf{y}_b||_2^2$.

b. (8 pts) Prove that $\mathbf{T}$ is a tight frame and show how $\mathbf{x}$ can be recovered from $\mathbf{T}^t\mathbf{x}$.

c. (10 pts) Let $\mathbf{x}$ denote a vector having an exact sparse representation with l_0 norm of 2 in the dictionary defined by the columns of $\mathbf{T}$. Prove that $||\mathbf{y}_a||_0 + ||\mathbf{y}_b||_0 \geq 4$, where $||\ ||_0$ represents the $l + 0$ norm.