EE 596 Wavelets (02/18/16)

Midterm #1

Name: ______________________________

Student ID Number: ____________________

The exam is open book and open notes. Show all your work in the exam pages. Results that are not explained or justified may not receive full credit even if they are correct. Time management is important, try to do as much as possible, so move on if you are having difficulties with one of the problems.

1 (40 points)	
2 (40 points)	
3 (20 points)	
Total (100 points)	

Problem 1 (40 pts). (Note: All questions can be solved independently)

Let E be the vector space of polynomials of a variable x with degree up to 3. One vector (polynomial) in this space is defined as $p(x) = a_0 + a_1x + a_2x^2 + a_3x^3$, where all $a_i \in \mathbb{R}$. The sum of two polynomials and multiplication by a scalar are defined in the usual way, i.e., $p(x) + q(x)$ and $\alpha p(x)$.

a. (10 pts) Define $p_0(x) = 1$, $p_1(x) = x$, $p_2(x) = x^2$ and $p_3(x) = x^3$. Prove that $\{p_0(x),\ p_1(x),\ p_2(x),\ p_3(x)\}$ forms a basis for E.

b. (15 pts) Are the following sets of vectors S_i subspaces of E? Justify your answers with a proof or a counter-example.

- $S_1 = \{q(x) \in E \ | \ q(x) = a_2 x^2 + a_3 x^3, \ \ a_2, a_3 \in \mathbb{R}\}$
- $S_2 = \{q(x) \in E \ | \ q(x) = 1 + a_1 x, \ \ a_1 \in \mathbb{R}\}$
- $S_2 = \{q(x) \in E \ | \ q(x) = a_0 q_0(x) + a_1 q_1(x), \ \ a_0, a_1 \in \mathbb{R}\}$, where $q_0(x)$ and $q_1(x)$, two polynomials in E, are given.

c. (15 pts) Define $q_0(x) = 1$, and then for $i = 1, \ldots, 3$, define $q_i(x) = \alpha x q_{i-1}(x) + \beta$. For which values of α and β does $\{q_0(x),\ q_1(x),\ q_2(x),\ q_3(x)\}$ form a bi-orthogonal basis for E. For $\beta = 0$, find the dual basis.

Problem 2 (40 pts). (Note: All questions can be solved independently)

Consider the space of discrete time finite energy sequences $l_2(\mathbb{Z})$. Let $\delta(n)$ denote the impulse function and for a given $\theta \in [0, \pi]$, define the set of vectors:
$S_\theta = \{\phi_{2k,\theta}(n), \phi_{2k+1,\theta}(n), k \in \mathbb{Z}, \}$ where

$$\phi_{2k,\theta}(n) = \alpha\delta(n-2k) + \beta\delta(n-(2k+1))$$

$$\phi_{2k+1,\theta}(n) = -\beta\delta(n-2k) + \alpha\delta(n-(2k+1))$$

with $\alpha = \cos(\theta), \beta = \sin(\theta)$

a. (15 pts) Prove that S_θ is an orthonormal basis for $l_2(\mathbb{Z})$.

b. (15 pts) Define an overcomplete set $S' = S_{\theta_1} \cup S_{\theta_2}$, $\theta_1 \neq \theta_2$. Prove that S' is a tight frame and provide the corresponding frame bounds.

c. (10 pts) For S' as defined above choose θ_1, θ_2 so that the resulting dictionary is best in terms minimizing the maximum coherence. Justify your answer.

Problem 3 (20 pts). (Note: All questions can be solved independently)

Let $\phi_0(n)$ be a real-valued function in $l_2(\mathbb{Z})$, such that $||\phi_0(n)||^2 = 1$, with time localization given by $\mu_t(\phi_0)$ and $\sigma_t^2(\phi_0)$, as defined in class.

$$\mu_t(\phi_0) = \sum_{n=-\infty}^{n=+\infty} n\phi_0(n)^2.$$

$$\sigma_t^2(\phi_0) = \sum_{n=-\infty}^{n=+\infty} (n-\mu_t)^2\phi_0(n)^2.$$

a. (10 pts) Denote $\phi_k(n) = \phi_0(n-k)$, prove that its time localization is the same as that of ϕ_0, that is $\sigma_t^2(\phi_k) = \sigma_t^2(\phi_0)$.

b. (10 pts) Define $\phi_0'(n)$ as follows: $\phi_0'(2n) = \phi_0(n)$ and $\phi_0'(2n+1) = 0$. Compute $\sigma_t^2(\phi_0')$ as a function of $\sigma_t^2(\phi_0)$. Is $\sigma_f^2(\phi_0')$ greater or smaller than $\sigma_f^2(\phi_0)$? Or you do not have enough information to say? Justify your answer.