MATH-545L: Assignment 2.5

Saket Choudhary
2170058637

1 Part a

$$\vec{x(t)} = \begin{bmatrix} y(t) & \dot{y(t)} \end{bmatrix}^T$$

$$\begin{aligned} m\ddot{y} + c\dot{y} + ky &= bu \\ \ddot{y} + \frac{c}{m}\dot{y} + \frac{k}{m}y &= \frac{b}{m}u \\ \ddot{y} &= -\frac{c}{m}\dot{y} - \frac{k}{m}y\frac{b}{m}u \end{aligned}$$

Thus we have:

$$\begin{aligned} \frac{dy}{dt} &= \dot{y} \\ \frac{d\dot{y}}{dt} = -\frac{c}{m}\dot{y} - \frac{k}{m}y &+ \frac{b}{m}u \end{aligned}$$

Thus,

$$\begin{aligned} \frac{d}{dt}\begin{bmatrix} y(t+1) \\ \dot{y}(t+1) \end{bmatrix} &= \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{c}{m} \end{bmatrix}\begin{bmatrix} y(t) \\ \dot{y}(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \\ \frac{d}{dt}\vec{x}(t+1) &= \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{c}{m} \end{bmatrix}\vec{x}(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) \\ \frac{d}{dt}\vec{x}(t+1) &= A\vec{x}(t) + Bu(t) \end{aligned}$$

and the equation for measurement is given by in the interval $[k\tau, (k+1)\tau)$:

$$\begin{aligned} y_k &= y(k\tau) + \omega_k \\ y_k &= \begin{bmatrix} 1 \\ 0 \end{bmatrix}\begin{bmatrix} y(t) \\ \dot{y}(t) \end{bmatrix} + \omega_k \\ y_k &= \begin{bmatrix} 1 \\ 0 \end{bmatrix}\vec{x}_k + \omega_k \end{aligned}$$

In the domain $[k\tau, (k+1)\tau)$ we re-formulate the equations as:

$$\begin{aligned} \frac{d}{dt}\vec{x}(t+1) &= A\vec{x}(t) + Bu_t \\ \text{where } A &= \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{c}{m} \end{bmatrix} \\ \text{and } B &= \begin{bmatrix} 0 \\ \frac{b}{m} \end{bmatrix} \end{aligned}$$

And the observation equation:

$$y_k = C\vec{x}_k + \omega_k$$
$$\text{where } C = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

and the initial condition is given by:

$$\vec{x}(0) = \begin{bmatrix} y(0) \\ \dot{y}(0) \end{bmatrix} = \begin{bmatrix} y_0 + \eta \\ y_1 + \varsigma \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \end{bmatrix} + \begin{bmatrix} \eta \\ \varsigma \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \end{bmatrix} + \vec{\epsilon}$$
$$\text{where } \vec{\epsilon} = \mathcal{N}(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, Q)$$
$$\text{and } Q = \begin{bmatrix} \alpha^2 & 0 \\ 0 & \beta^2 \end{bmatrix}$$
$$\text{where } \eta \sim \mathcal{N}(0, \alpha^2)$$
$$\text{and } \varsigma \sim \mathcal{N}(0, \beta^2)$$

We now use variation of parameters on the interval $[k\tau, (k+1)\tau)$ t obtain:

$$\vec{x}_{k+1} = \exp(A\tau)x_k + \int_{k\tau}^{k\tau+\tau} \exp(A(s - k\tau))B(s)u(s)ds$$
$$\vec{x}_{k+1} = \exp(A\tau)x_k + \int_{k\tau}^{k\tau+\tau} \exp(A(s - k\tau))dsB(u_k + \epsilon_k)$$
$$\vec{x}_{k+1} = \exp(A\tau)x_k + \int_{0}^{\tau} \exp(As')ds'B(u_k + \epsilon_k) \text{ where } s' = s + k\tau$$
$$\vec{x}_{k+1} = \exp(A\tau)x_k + \int_{0}^{\tau} \exp(As')ds'B(u_k + \epsilon_k)$$
$$\vec{x}_{k+1} = \exp(A\tau)x_k + A^{-1}(\exp(A\tau) - I)Bu_k + A^{-1}(\exp(A\tau) - I)B\epsilon_k$$

Now,

$$\exp(A\tau) = I + \frac{A\tau}{1!} + \frac{A^2\tau^2}{2!} + \frac{A^3\tau^3}{3!} + \dots$$

2 Part b

$$\vec{x}_{k+1} = \phi\vec{x}_k + \psi u_k + \vec{v}_k$$

The cofficients of the above equation are given by:

$$A = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{c}{m} \end{bmatrix}$$

$$\text{and } B = \begin{bmatrix} 0 \\ \frac{b}{m} \end{bmatrix}$$

$$\phi = \exp(A\tau)$$

$$\psi = A^{-1}(\exp(A\tau) - I)B$$

$$\vec{v}_k = A^{-1}(\exp(A\tau) - I)B\epsilon_k = \psi\epsilon_k$$

And thus $\vec{v}_k \sim \mathcal{N}(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, V)$ where $V = \sigma^2\psi\psi^T$ where $\psi_{2\times1} = A^{-1}(\exp(A\tau) - I)B$

3 Part c

$$\begin{aligned} x_{k+1} &= \phi x_k + \psi u_k + v_k \\ \bar{x}_{k+1} &= \phi \bar{x_k} + \psi u_k \\ x_{k+1} - \bar{x}_{k+1} &= \phi(x_k - \bar{x_k}) + v_k \end{aligned}$$

$$\begin{aligned} P_{k+1|k} &= E((x_{k+1} - \bar{x}_{k+1})(x_{k+1} - \bar{x}_{k+1})^T) \\ &= E[(\phi(x_k - \bar{x_k}) + v_k)((x_k - \bar{x_k})^T\phi^T + v_k^T)] \\ &= E[\phi(x_k - \bar{x_k})(x_k - \bar{x_k})^T\phi^T + 2(x_k - \bar{x_k})\phi v_k + v_k v_k^T] \\ &= \phi P_{k|k}\phi^T + V_k \end{aligned}$$

Prediction:

$$\begin{aligned} \hat{x}_{k+1|k} &= \phi\hat{x}_k + \psi u_k \\ P_{k+1|k} &= \phi P_{k|k}\phi^T + V_k \end{aligned}$$

Observed/Measured:

$$\begin{aligned} x_{k+1|k} &= \phi x_k + \psi u_k + \vec{v}_k \\ y_{k+1|k} &= Cx_{k+1} + \omega_k \end{aligned}$$

Assume update step to be linear combination of prediction and observation. We need to obtain K_1, K_2 such that the estimator is unbiased and is minimum variance estimate.

$$\hat{x}_{k+1|k+1} = K_1\hat{x}_{k+1|k} + K_2 y_{k+1|k}$$

Define error $e_{k+1|k+1} = \hat{x}_{k+1|k+1} - x_{k+1}$

We look for unbiased estimator $E[e_{k+1|k+1}] = 0$
Thus,

$$\begin{aligned}
e_{k+1|k+1} &= \hat{x}_{k+1|k+1} - x_{k+1} \\
&= K_1\hat{x}_{k+1|k} + K_2(Cx_{k+1} + \omega_{k+1}) - x_{k+1} \\
&= K_1\hat{x}_{k+1|k} + (K_2C - I)x_{k+1} + K_2\omega_{k+1}
\end{aligned}$$

Now,

$$\begin{aligned}
&E[e_{k+1|k+1}] = 0 \\
&\implies K_1 = I - K_2C
\end{aligned}$$

Thus,

$$\begin{aligned}
\hat{x}_{k+1|k+1} &= (I - K_2C)\hat{x}_{k+1|k} + K_2y_{k+1|k} \\
&= (I - K_2C)\hat{x}_{k+1|k} + K_2(Cx_{k+1} + \omega_{k+1}) \\
&= \hat{x}_{k+1|k} + K_2C(x_{k+1} - \hat{x}_{k+1|k}) + \omega_{k+1}
\end{aligned}$$

Now,

$$\begin{aligned}
e_{k+1|k+1} &= \hat{x}_{k+1|k+1} - x_{k+1} \\
&= K_1\hat{x}_{k+1|k} + K_2(Cx_{k+1} + \omega_{k+1}) - x_{k+1} \\
&= (I - K_2C)\hat{x}_{k+1|k} + (K_2C - I)x_{k+1} + K_2\omega_{k+1} \\
&= (I - K_2C)(\hat{x}_{k+1|k} - x_{k+1}) + K_2\omega_{k+1} \\
&= (I - K_2C)e_{k+1|k} + K_2\omega_{k+1}
\end{aligned}$$

where $e_{k+1|k} = \hat{x}_{k+1|k} - x_{k+1}$ Then,

$$\begin{aligned}
P_{k+1|k+1} &= E[e_{k+1|k+1}e_{k+1|k+1}^T] \\
&= E[((I - K_2C)e_{k+1|k} + K_2\omega_{k+1})((I - K_2C)e_{k+1|k} + K_2\omega_{k+1})^T] \\
&= (I - K_2C)E[e_{k+1|k}^T e_{k+1|k}](I - K_2C)^T + K_2E[\omega_{k+1}\omega_{k+1}^T]K_2^T \\
&= (I - K_2C)P_{k+1|k}(I - K_2C)^T + K_2W_kK_2^T \\
&= (P_{k+1|k} - K_2CP_{k+1|k})(I - K_2C)^T + K_2W_kK_2^T \\
&= P_{k+1|k} - K_2CP_{k+1|k} - P_{k+1|k}C^TK_2^T + K_2CP_{k+1|k}C^TK_2^T + K_2W_kK_2^T \\
&= P_{k+1|k} - K_2CP_{k+1|k} - P_{k+1|k}C^TK_2^T + K_2(CP_{k+1|k}C^T + W_k)K_2^T
\end{aligned}$$

To minimize $\frac{\partial P_{k+1|k+1}}{\partial K_2} = 0$

$$\begin{aligned}
\frac{\partial P_{k+1|k+1}}{\partial K_2} &= -2P_{k+1|k}^TC + 2K_2(CP_{k+1|k}C^T + W_k) = 0 \\
&\implies K_2 = P_{k+1|k}^TC(CP_{k+1|k}C^T + W_k)^{-1}
\end{aligned}$$

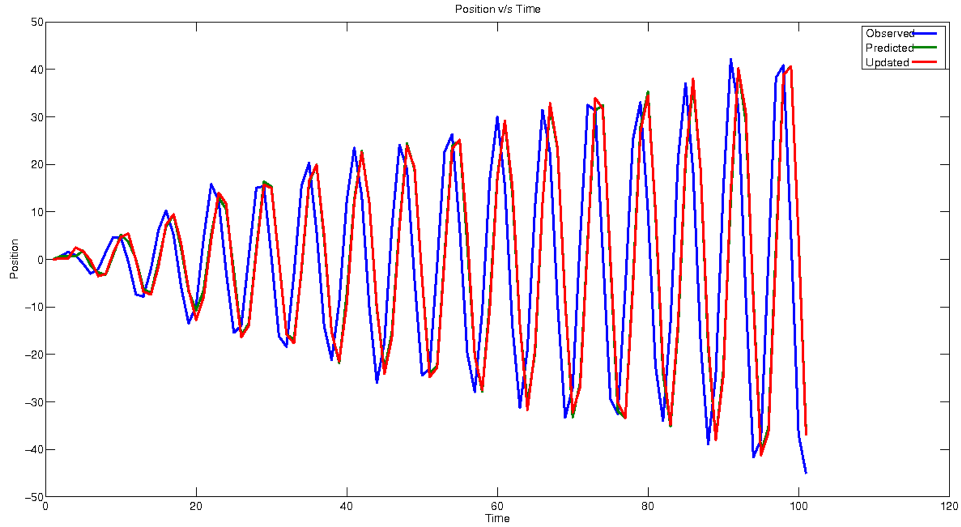

Figure 1: Measured, predicted, filtered position with damping and resonance

Prediction:

$$\hat{x}_{k+1|k} = \phi \hat{x}_k + \psi u_k$$
$$P_{k+1|k} = \phi P_{k|k} \phi^T + V_k$$

Observed/Measured:

$$x_{k+1|k} = \phi x_k + \psi u_k + \vec{v}_k$$
$$y_{k+1|k} = C x_{k+1} + \omega_k$$

Update Step:

$$\hat{x}_{k+1|k+1} = \hat{x}_{k+1|k} + K_2(y_k - C\hat{x}_{k+1|k})$$
$$P_{k+1|k+1} = P_{k+1|k} - K_2 C P_{k+1|k}$$
$$K_2 = P^T_{k+1|k} C (C P_{k+1|k} C^T + W_k)^{-1}$$

4 Part d

For $m = 100, c = 1, k = 1, b = 1, Var(\omega) = 0.1, Var(\epsilon) = 0.1$
For $m = 1, c = 1, k = 1, b = 1, Var(\omega) = 0.1, Var(\epsilon) = 0.1$
For $m = 1, c = 1, k = 1, b = 1, Var(\omega) = 2, Var(\epsilon) = 2$

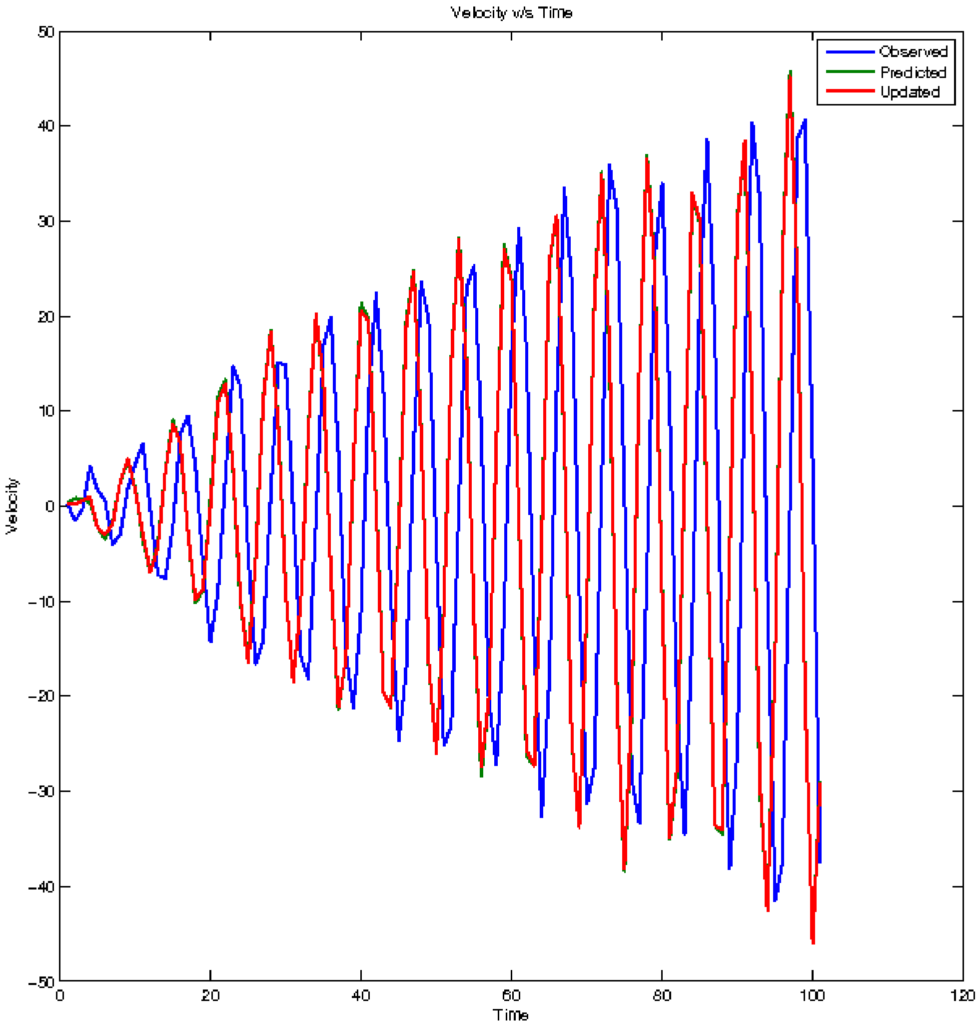

Figure 2: Measured, predicted, filtered velocity with damping and resonance

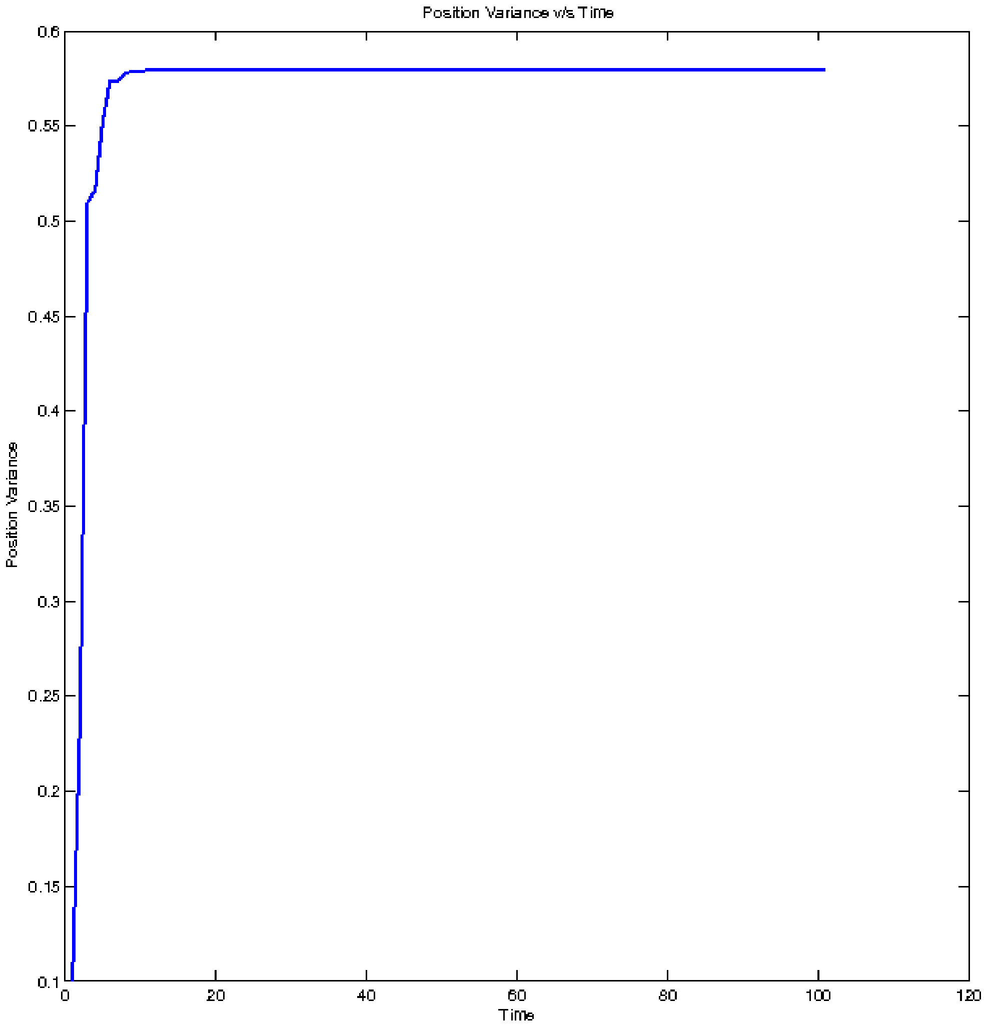

Figure 3: Position variance with damping and resonance

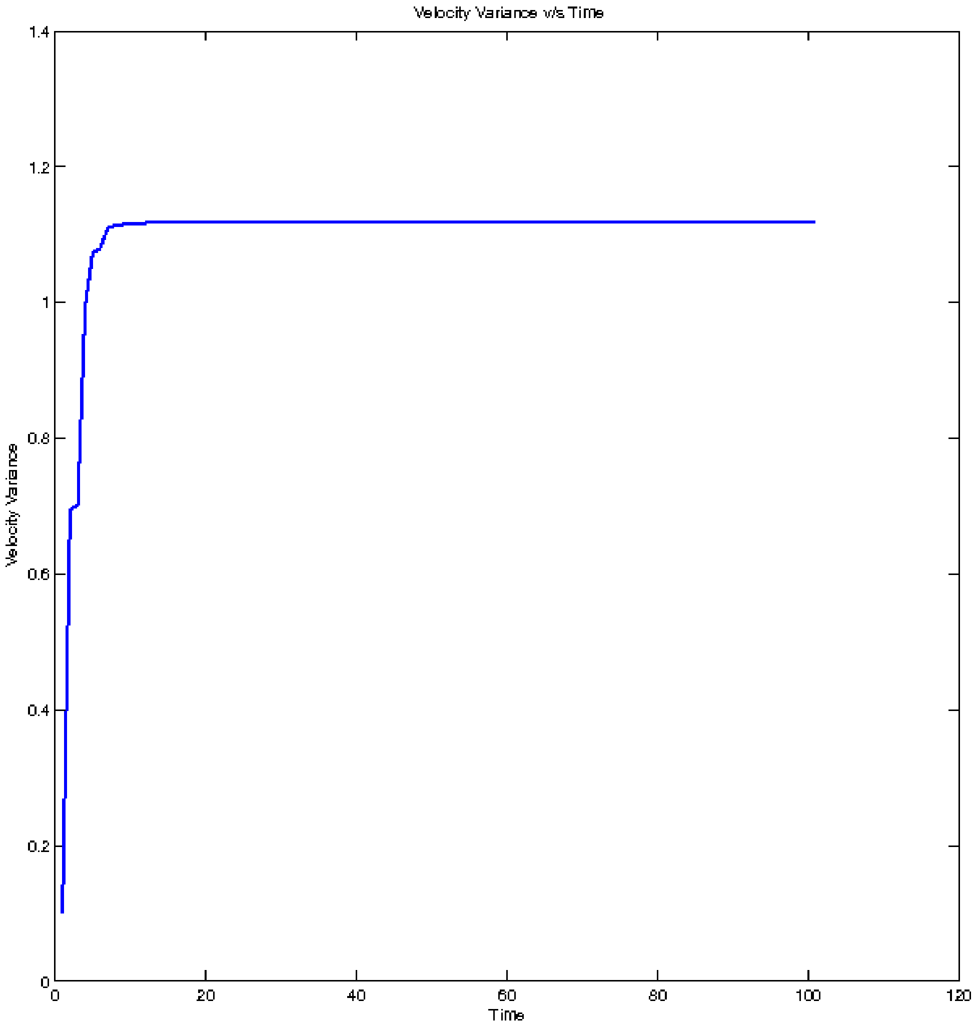

Figure 4: Velocity variance with damping and resonance

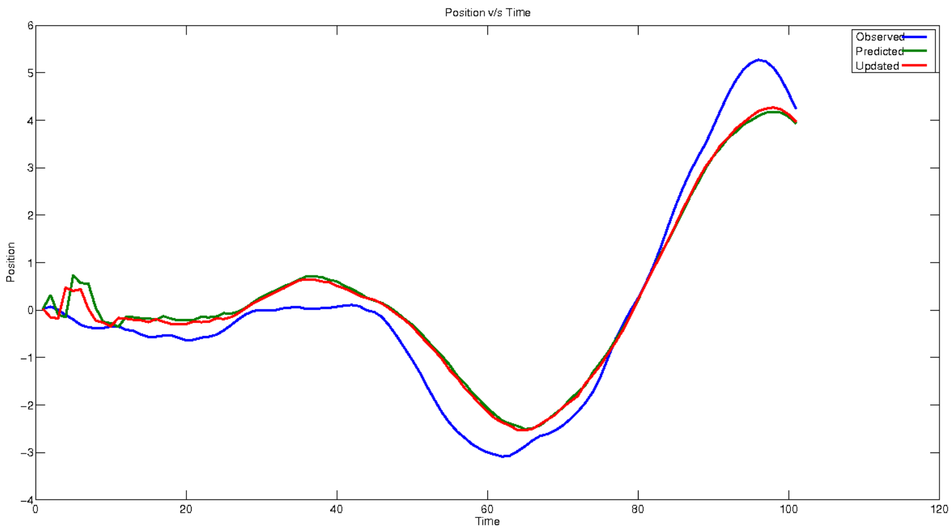

Figure 5: Measured, predicted, filtered position with high mass

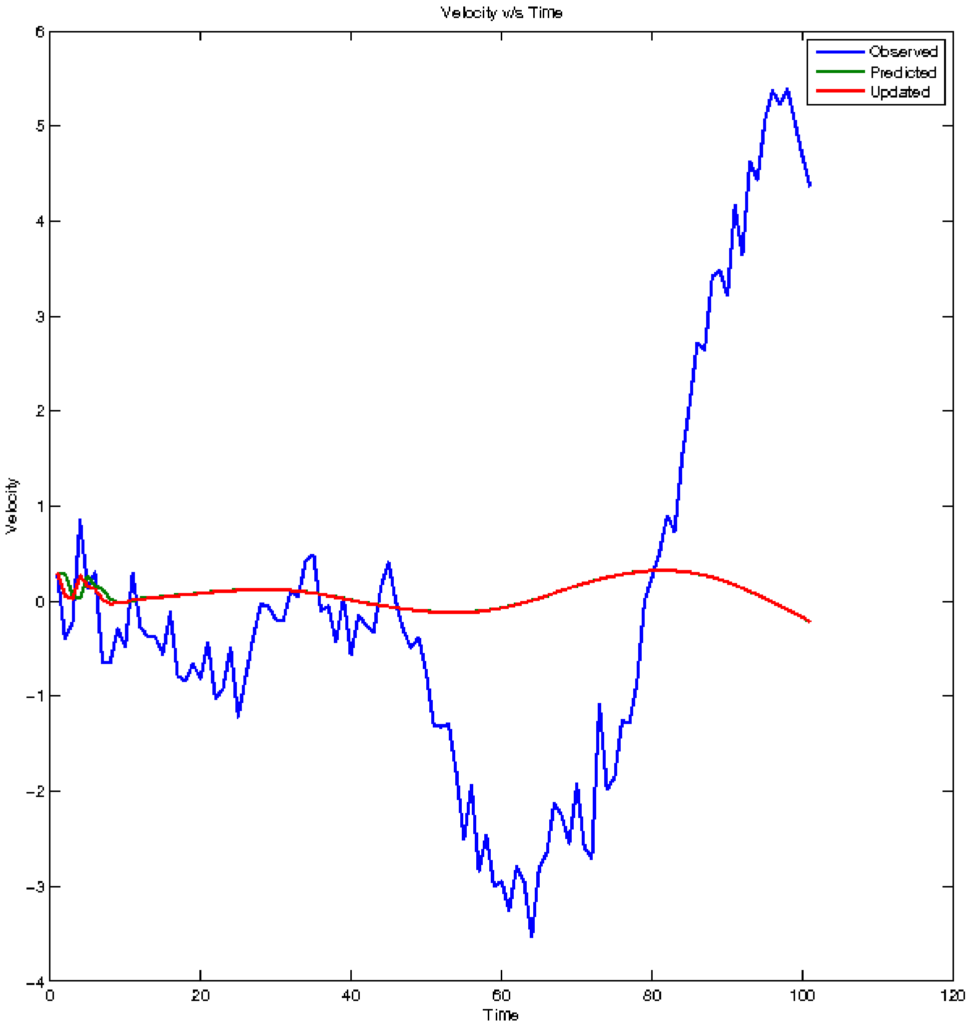

Figure 6: Measured, predicted, filtered velocity with high mass

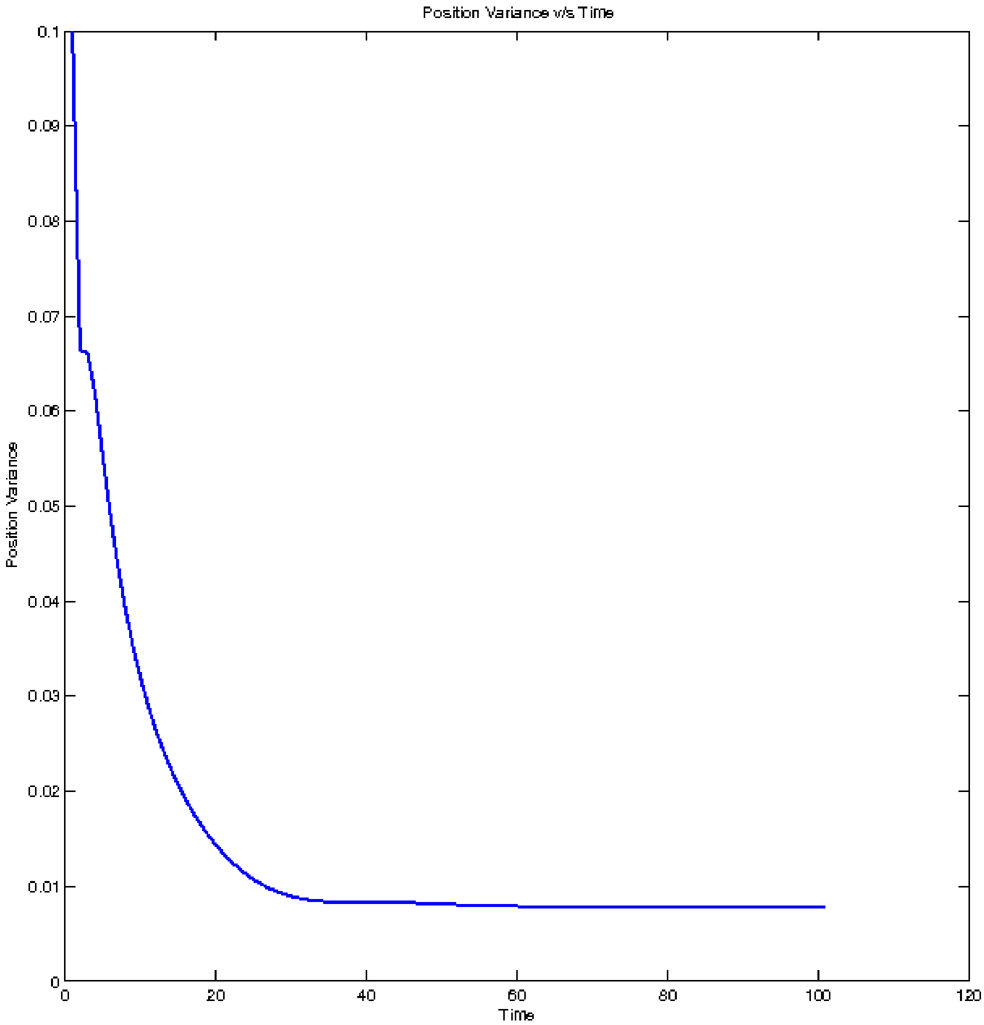

Figure 7: Position variance with high mass

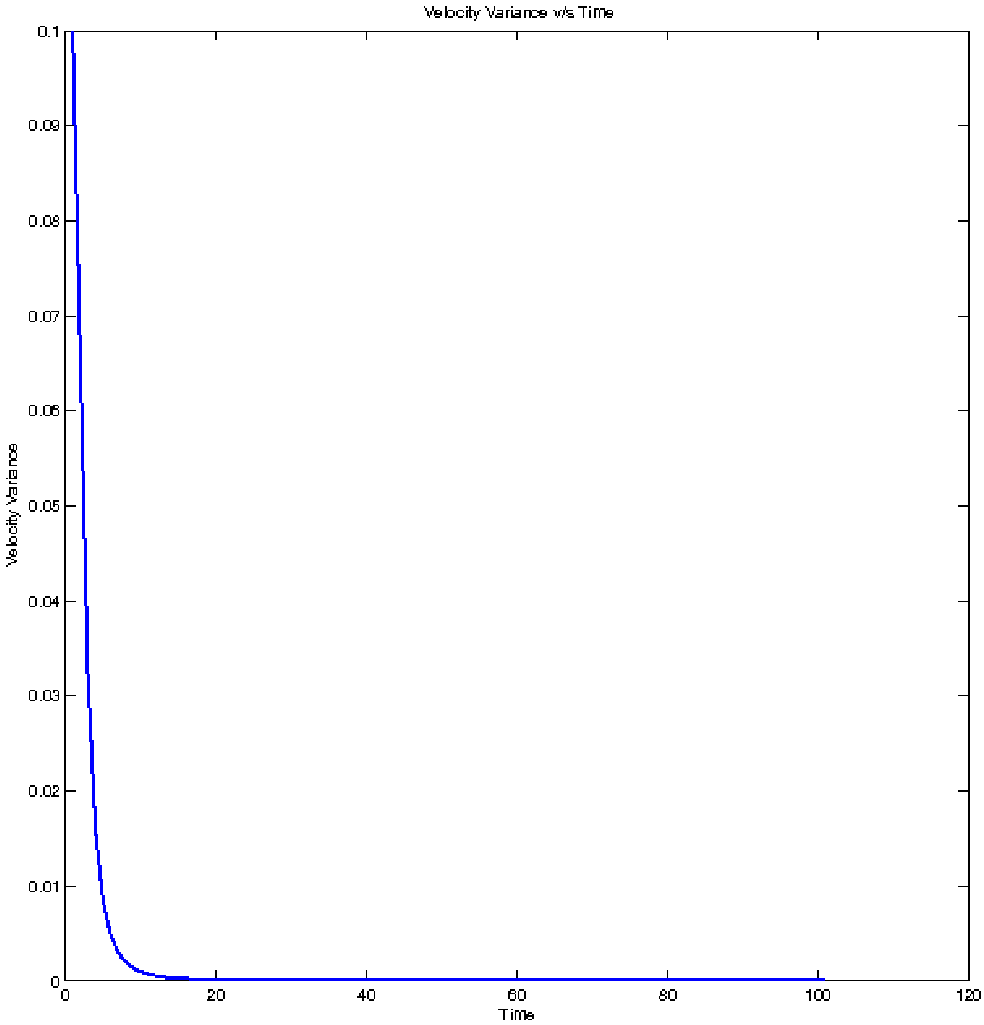

Figure 8: Velocity variance with high mass

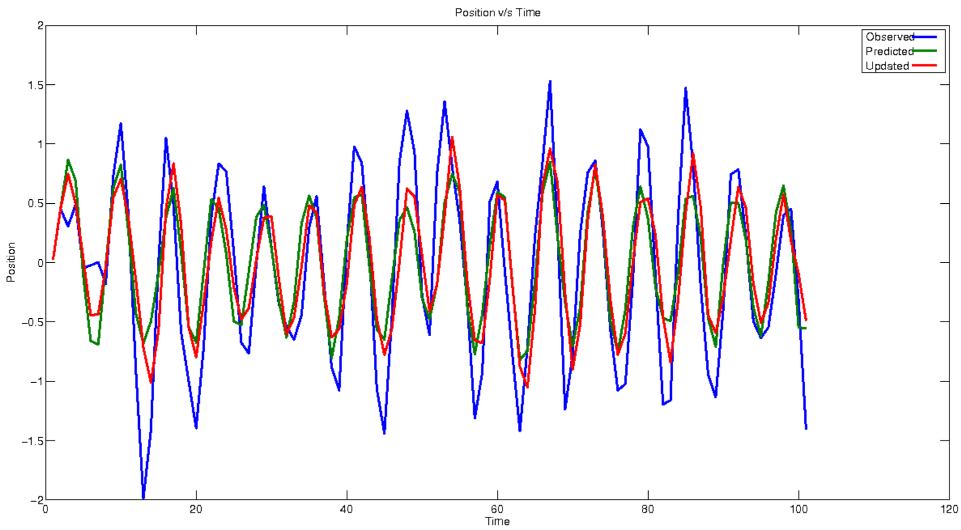

Figure 9: Measured, predicted, filtered position with low mass

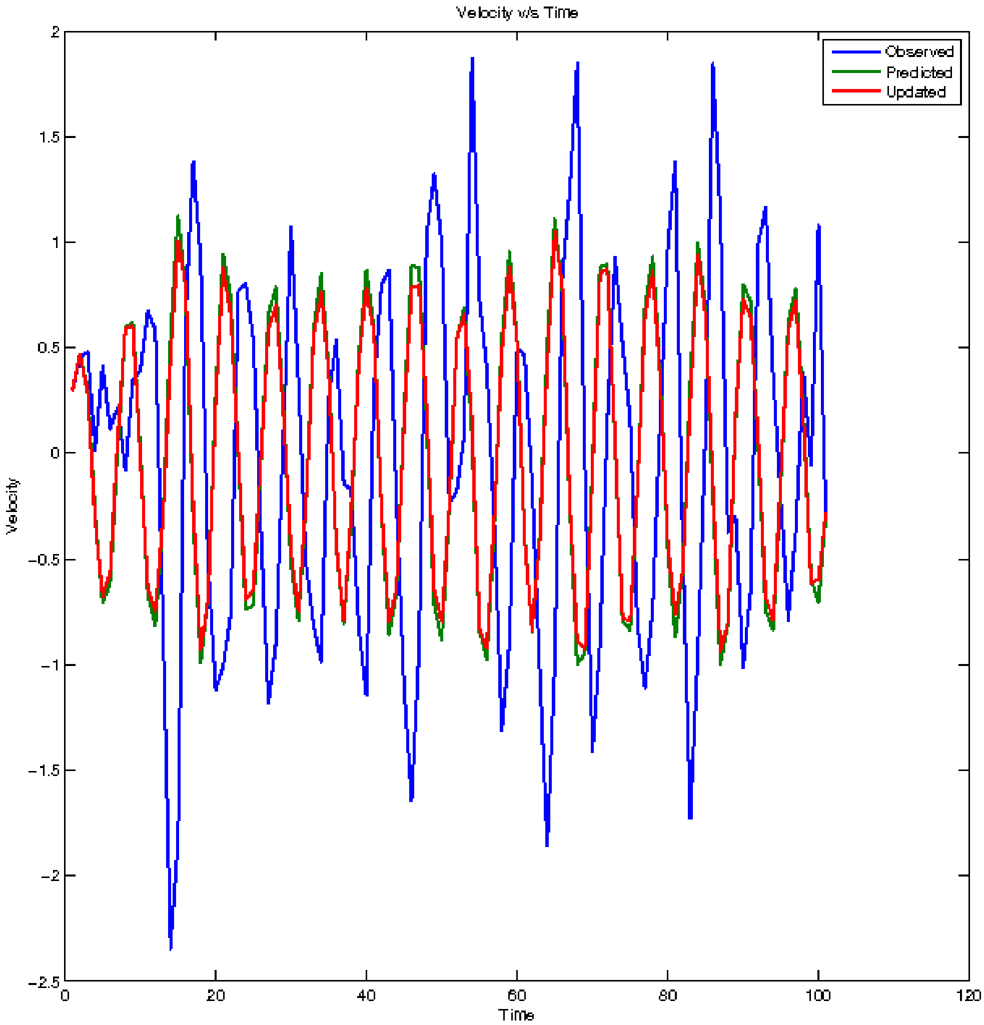

Figure 10: Measured, predicted, filtered velocity with low mass

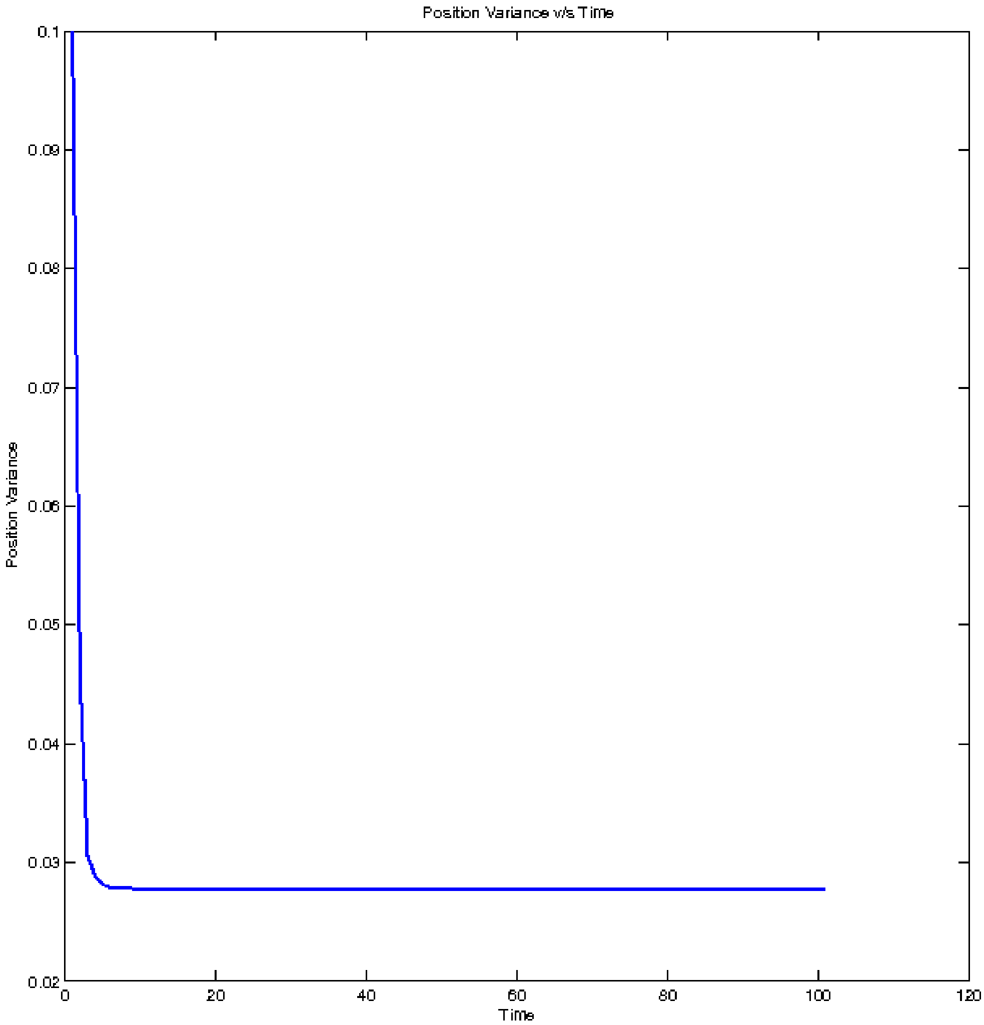

Figure 11: Position variance with low mass

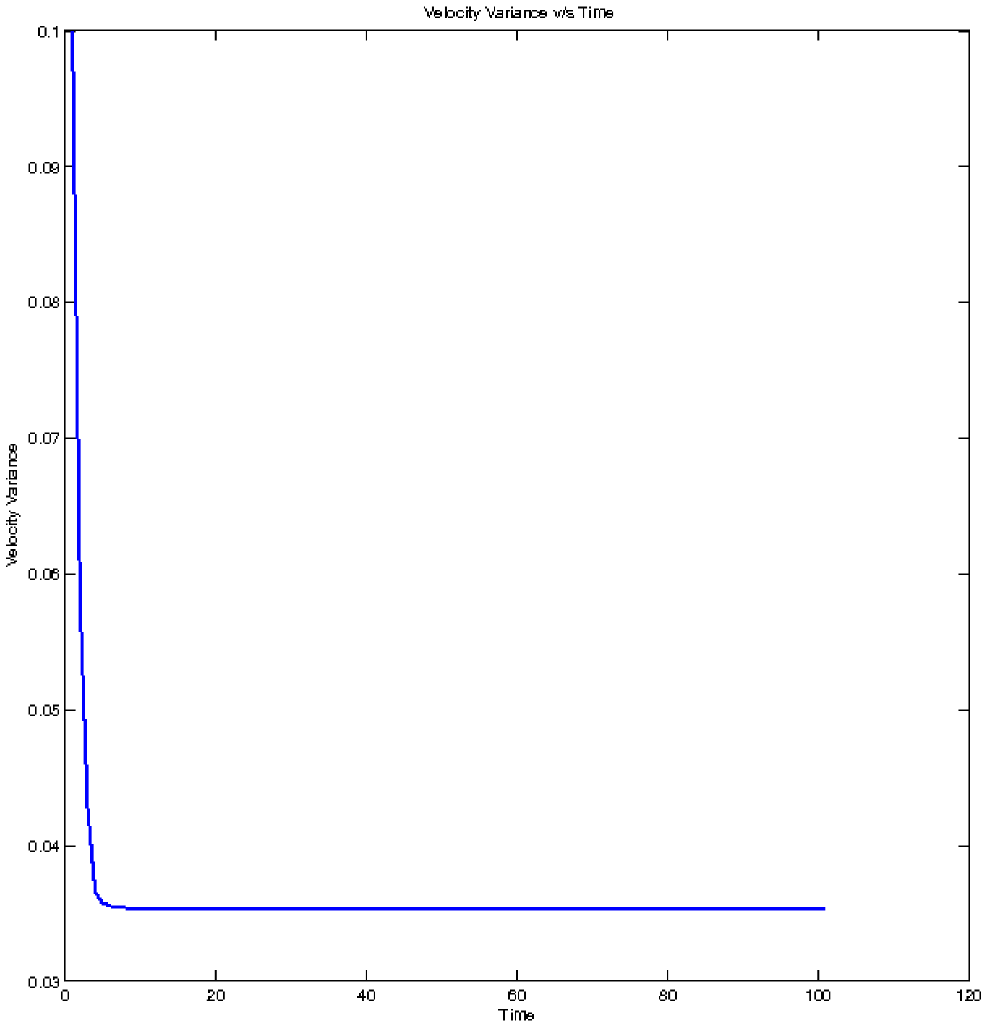

Figure 12: Velocity variance with low mass

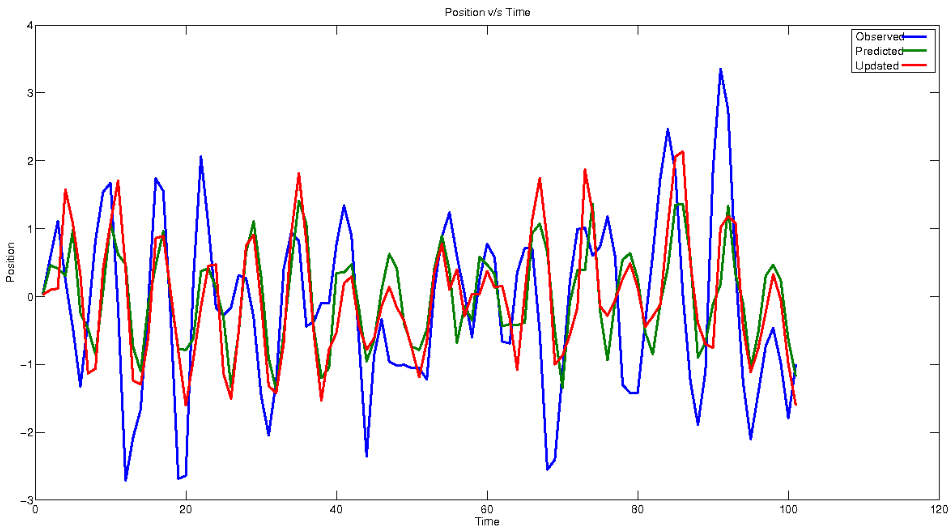

Figure 13: Measured, predicted, filtered position with low mass high variance

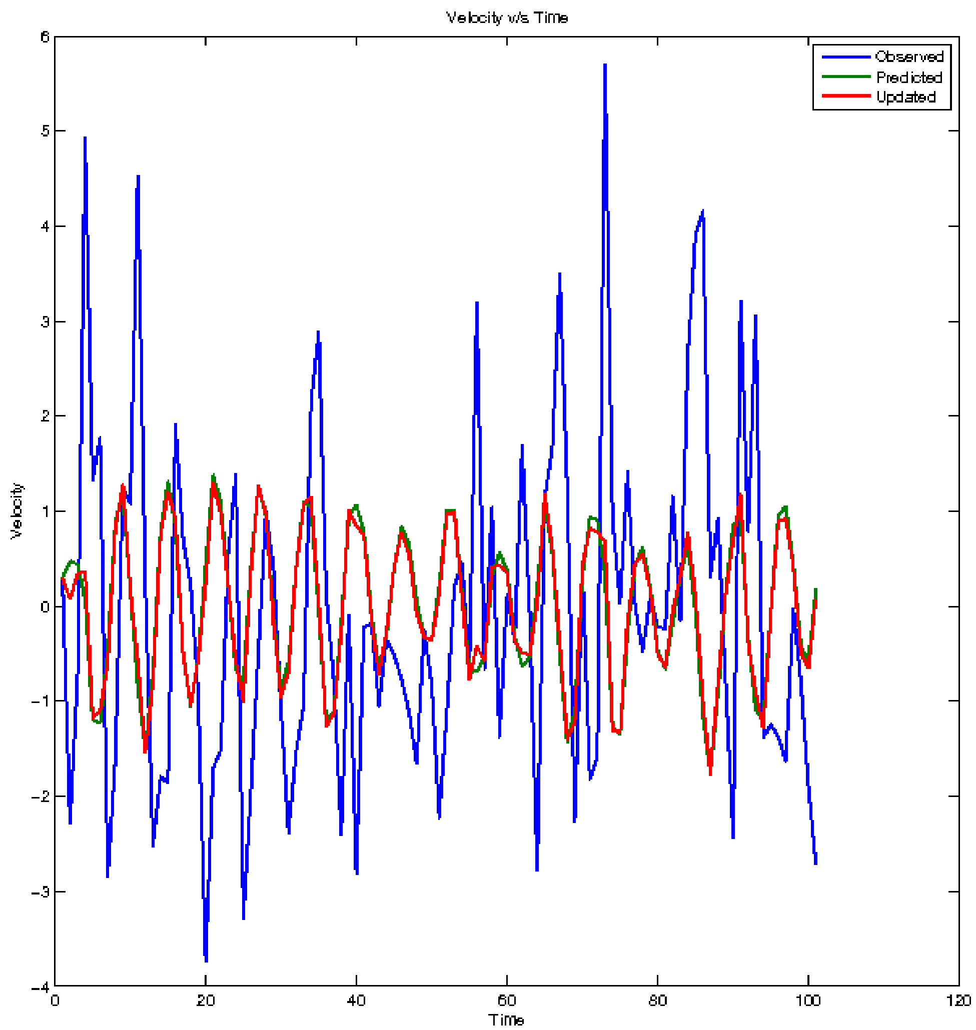

Figure 14: Measured, predicted, filtered velocity with low mass high variance

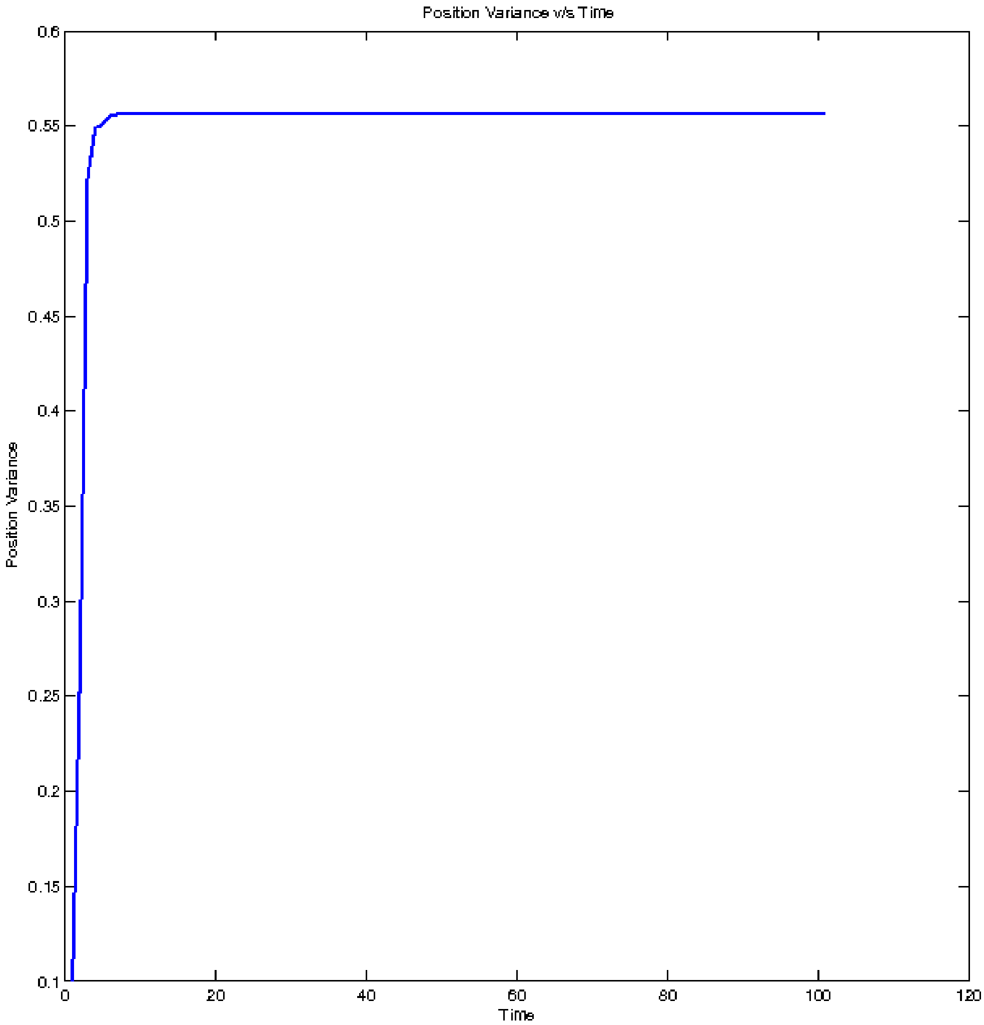

Figure 15: Position variance with low mass high variance

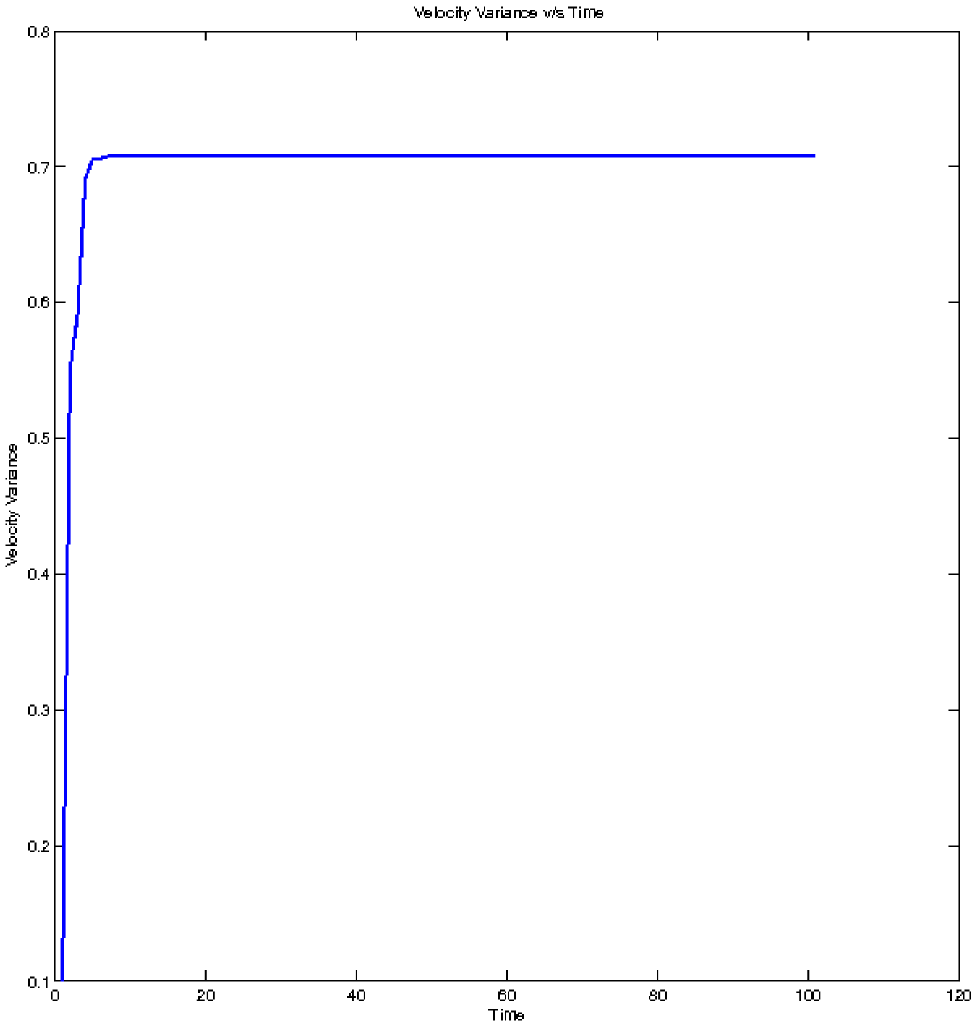

Figure 16: Velocity variance with low mass high variacne