EE-588: Homework # 1

Due on Tuesday, September 25, 2019

Saket Choudhary
2170058637

Contents

2.5

The distance beween hyperplances $\{x \in \mathbb{R}^n | a^T x = b_1\}$ and $\{x \in \mathbb{R}^n | a^T x = b_2\}$ is given by the distance between the points of intersection of the normal a^T with its points of intersection at these two hyperplanes respectively. The points of intersection of a line from the origin to the hyperplane that is parallel to a is given by $x_1 = \frac{b_1}{||a||_2^2} a$ and $x_2 = \frac{b_2}{||a||_2^2} a$ respectively. The corresponding distance between them is:

$$\begin{aligned}\text{dist}(x_1, x_2) &= ||x_1 - x_2||_2 \\ &= \frac{|b_1 - b_2|}{||a||_2^2} ||a|| \\ &= \frac{|b_1 - b_2|}{||a||_2}\end{aligned}$$

2.7

Let $S = \{x \mid ||x - a||_2 \leq ||x - b||_2\}$

$$\begin{aligned} ||x - a||_2 &\leq ||x - b||_2 \\ \implies ||x - a||_2^2 &\leq ||x - b||_2^2 \\ (x - a)^T (x - a) &= (x - b)^T (x - b) \\ 2(b - a)^T x &\leq b^T b - a^T \\ s^T x &\leq t \quad \text{where } s = 2(b - a); t = b^T b - a^T a \end{aligned}$$

$s^T x \leq t$ is a halfspace. See below figure for interpretation.

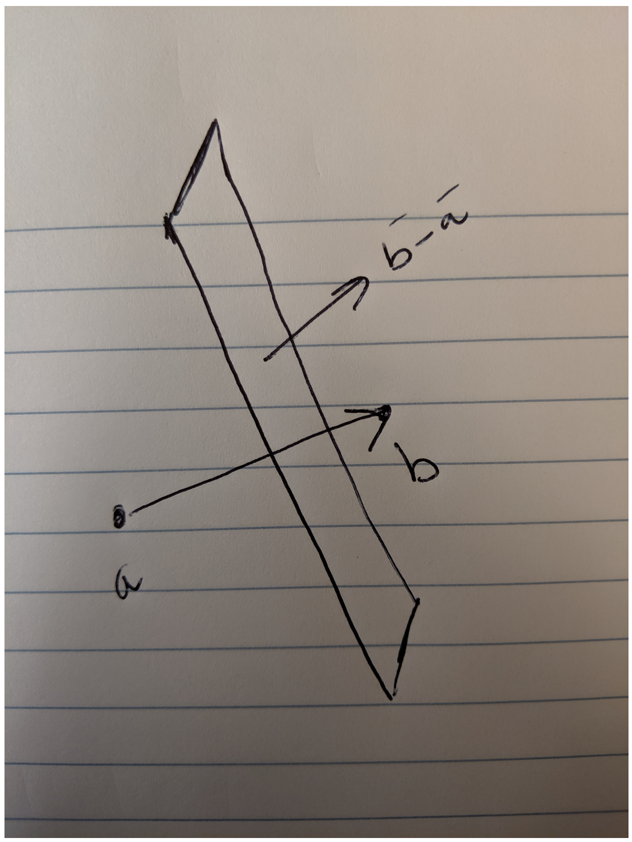

Figure 1: Problem 2.7. Geometric interpretation> The hyperplane has a normal in the direction of $b - a$ and contains points that are equidistant from both a and b.

2.12

(a) Slab is intersection of two half space $a^Tx \leq \beta$ and $a^Tx \geq \alpha$ which are both convex. Hence, $\boxed{\text{Convex}}$.

(b) Similar to part (a), rectangle is alos intersection of two convex sets and hence $\boxed{\text{Convex}}$.

(c) Similar to parts (a) and (b), wedge is intersection of two halfspaces $a_1^Tx \leq b_1$ and $a_2^Tx \leq b_2$ and hence $\boxed{\text{Convex}}$.

(d) for a fixed y, $||x - x_0|| \leq ||x - y||$ is a convex set. The set of close points closer to a given point x_0 than the set is given by the intersection of sets $\cap_{y \in S}\{x \mid ||x - x_0|| \leq ||x - y||\}$ and hence $\boxed{\text{Convex}}$.

(e) Take a simple set $S = \{-1, 1\}$ and a single point set $T = \{0\}$. now the set $\{x|\text{dist}(x, S) \leq \text{dist})(x, T)$ is given by $x \geq 1/2$ or $x \leq -1/2$ which is non-convex. hence $\boxed{\text{Non-Convex}}$.

(f) $x + S_2 \subseteq S_1 \iff x + s_2 \subseteq S_1 \forall s_2 \in S_2$.

$$\begin{aligned}
&\{x|x + S_2 \subseteq S_1\} \\
&\quad = \cap_{s_2 \in S_2}\{x|x + s_2 \subseteq S_1\} \\
&\quad = \cap_{s_2 \in S_2}\{x|x \subseteq S_1 - s_2\} \\
&\quad = \cap_{s_2 \in S_2}\{S_1 - s_2\}
\end{aligned}$$

Thus $\{x|x + S_2 \subseteq S_1\}$ is an intersection of convex set $S_1 - s_2$ and hence $\boxed{\text{Convex}}$.

(g)

$$\begin{aligned}
&\{x|||x - a||_2 \leq \theta||x - b||_2\} \\
&\quad = \{x|||x - a||_2^2 \leq \theta^2||x - b||_2^2\} \\
&\quad = \{x|(x - a)^T(x - a) \leq \theta^2(x - b)^t(x - b)\} \\
&\quad = \{x|x^Tx + a^Ta - 2a^Tx \leq \theta^2x^Tx + b^Tb - 2b^Tx\} \\
&\quad = \{x|(1 - \theta^2)x^Tx - 2(a^T - \theta^2b^T)x + a^Ta - \theta^2b^Tb \leq 0\}
\end{aligned}$$

This is rewritten as an euclidean ball

$$\{x|(x - c)^T(x - c) \leq r^2\}$$

$$c = \frac{a - \theta^2 b}{1 - \theta^2}$$

$$r = \sqrt{\left(\frac{\theta^2||b||_2^2 - ||a||_2^2}{1 - \theta^2} - ||c||_2^2\right)}$$

Hence, $\boxed{\text{Convex}}$.

2.28

$n = 1$: $x_1 \geq 0$

$n = 2$:

$$v^T X v = \begin{bmatrix} v_1 & v_2 \end{bmatrix} \begin{bmatrix} x_1 & x_2 \\ x_2 & x_3 \end{bmatrix} \begin{bmatrix} v_1 & v_2 \end{bmatrix}$$
$$= x_1 v_1^2 + 2x_2 v_1 v_2 + x_3 v_2^2$$
$$= x_1 \frac{v_1}{v_2}^2 + 2x_2 \frac{v_1}{v_2} + x_3$$
$$= x_1 t^2 + 2x_2 t + x_3 \quad \text{Let } t = \frac{v_1}{v_2} \text{ when } v_2 \neq 0$$

Since $v^T X v \geq 0$, we have $x_1 t^2 + 2x_2 t + x_3 \geq 0 \; \forall t$. Also when $v_2 = 0, v^T X v = x_1 v_1^2$ and hence $x_1 \geq 0$. Now, for $x_1 t^2 + 2x_2 t + x_3 \geq 0$, we have

$$4x_2^2 \leq 4x_1 x_3$$

or $x_1 x_3 \geq x_2^2$. We already have $x_1 \geq 0 \implies x_3 \geq 0$.

$$\boxed{x_1 \geq 0, x_3 \geq 0, x_1 x_3 \geq x_2^2}$$

$n = 3$:

$$v^X v = x_1 v_1^2 + 2x_2 v_1 v_2 + 2x_3 v_1 v_3 + x_4 v_2^2 + 2x_5 v_2 v_3 + x_6 v_3^2$$

If $x_1 = 0$ then, $x_2 = 0$ and $x_3 = 0$. Also from $n = 2$ case for $X \succcurlyeq 0$, we have

$$\begin{bmatrix} x_4 & x_5 \\ x_5 & x_6 \end{bmatrix} \succcurlyeq 0$$

which requires

$$x_4 \geq 0; x_6 \geq 0; x_4 x_5 \geq x_6^2$$

.

When $x_1 \neq 0$, we re write it as:

$$v^T X v = x_1 (v_1 + \frac{x_2}{x_1} v_2 + \frac{x_3}{x_1} v_3)^2 + (x_4 - \frac{x_2^2}{x_1}) v_2^2 + 2(x_5 - \frac{x_2 x_3}{x_1}) v_2 v_3 + (x_6 - \frac{x_3^2}{x_1}) z_3^2$$

2.28 continued on next page. . .

For $v^TXv \geq 0$, $x_1 \geq 0$. Also,

$$\begin{bmatrix} x_4 - \frac{x_2^2}{x_1} & x_5 - \frac{x_2x_3}{x_1} \\ x_5 - \frac{x_2x_3}{x_1} & x_6 - \frac{x_3^2}{x_1} \end{bmatrix} \succcurlyeq 0$$

which following the $n = 2$ case requires:

$$x_4 - \frac{x_2^2}{x_1} \geq 0; x_6 - \frac{x_3^2}{x_1} \geq 0; (x_4 - \frac{x_2^2}{x_1})(x_6 - \frac{x_3^2}{x_1}) \geq (x_5 - \frac{x_2x_3}{x_1})^2$$

In totality,

$$x_1 \geq 0$$
$$x_2 \geq 0$$
$$x_3 \geq 0$$
$$x_4x_1 \geq x_2^2$$
$$x_6x_1 \geq x_3^2$$
$$x_4x_6 \geq x_5^2$$
$$(x_4 - \frac{x_2^2}{x_1})(x_6 - \frac{x_3^2}{x_1}) \geq (x_5 - \frac{x_2x_3}{x_1})^2$$

3.2

The first curve cannot be convex as along the marked directions in Figure 1, it is not convex. It is quasiconvex as the sublevels are convex.
The second curve appears to be concave.

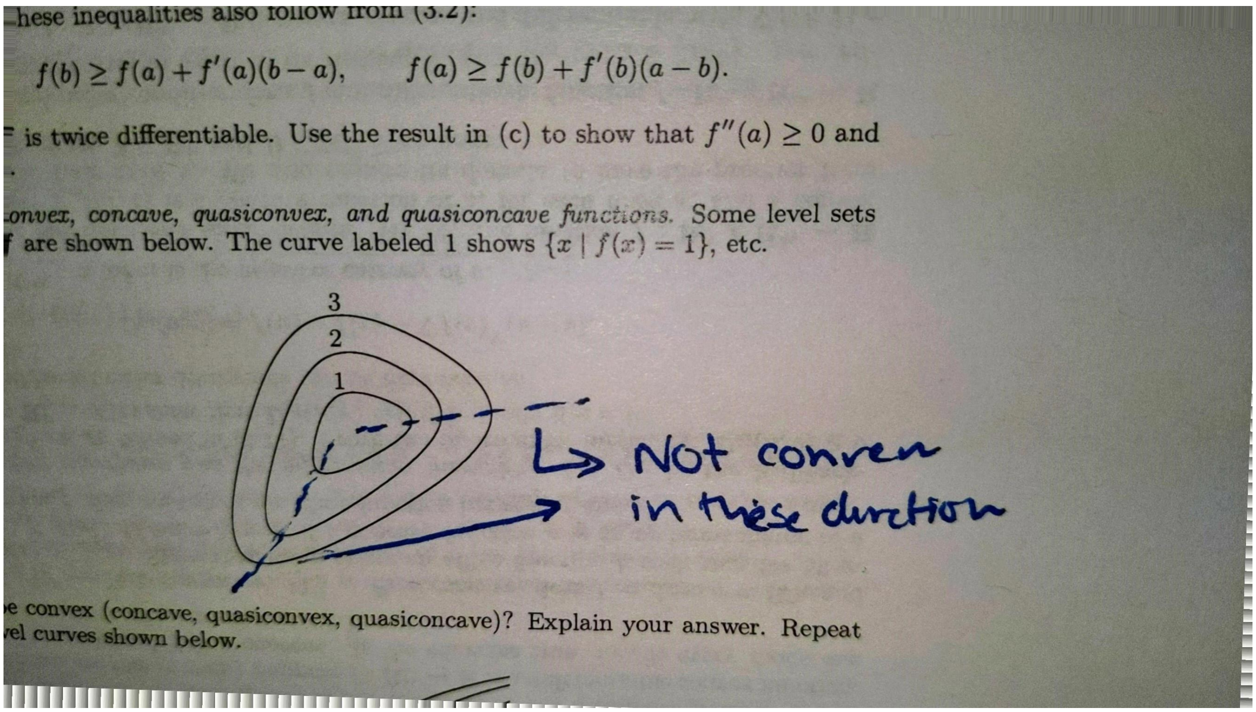

Figure 2: Problem 3.2

3.3

Since $f : \mathbb{R} \longrightarrow \mathbb{R}$ iis increasing and convex on (a, b), its inverse $g = f^{-1}$ will also be increasing in $(f(a), f(b))$. This can be proved as follows:
For $a < x < b$ and $a < y < b$, let $x < y$ without any loss of generality AND as $f : \mathbb{R} \longrightarrow \mathbb{R}$, i.e. the range and domain is $\mathbb{R}$.

$$x < y \iff f(x) < f(y)$$
$$g(x) < g(y) \iff g(f(x)) < g(f(y))$$

Hence $x < y \iff g(x) < g(y)$.
Consider $x, y \in \mathbb{R}$ and $0 \leq \theta \leq 1$. Define:

$$s :- g(\theta u + (1-\theta)v)$$
$$t :- \theta g(u) + (1-\theta)g(v)$$

Now, since $f(x)$ is convex, for :

$$\begin{aligned} f(s) &= f(g(\theta u + (1-\theta)v)) \\ &= \theta u + (1-\theta)v \\ &= \theta f(g(u)) + (1-\theta)f(g(v)) \\ &\geq f(\theta g(u) + (1-\theta)g(v)) && \text{Since } f(x) \text{ is convex} \\ &= f(t) \end{aligned}$$

As f is increasing in $\mathbb{R}$, $f(s) \geq f(t) \iff s \geq t$, i.e.,

$$g(\theta u + (1-\theta)v \geq \theta g(u) + (1-\theta)g(v)$$

Thus $g(x) : \mathbb{R} \longrightarrow \mathbb{R}$ is Concave.

Additional Exercise

Let $S = \{a \in \mathbb{R}^k | p(0) = 1, |p(t)| = 1 \forall \alpha \leq t \leq \beta\}$

$$\{a \in \mathbb{R}^k | p(0) = 1, |p(t)| = 1 \forall \alpha \leq t \leq \beta\} = \{a \in \mathbb{R}^k | p(0) = 1\} \cap \{a \in \mathbb{R}^k || p(t)| = 1 \forall \alpha \leq t \leq \beta\}$$

Define set $S_1 = \{a \in \mathbb{R}^k | p(0) = 1\}$ and $S_2^t = \{a \in \mathbb{R}^k || p(t)| = 1 \forall \alpha \leq \beta\}$. S_1 contains all points $a \in \mathbb{R}^k | p(0) = 1\}$ with the first component $a_1 = 1$ which is convex.
Set S_2 can be further decomposed as follows:

$$\{a \in \mathbb{R}^k || p(t)| = 1 \forall \alpha \leq \beta\} = \cap_{t \in [\alpha, \beta]} \{a \in \mathbb{R}^k || p(t)| \leq 1\}$$

From 2.12, $\{a \in \mathbb{R}^k || p(t)| \leq 1\}$ is a slab and convex and intersection of convex sets is convex, making $S_2^{(t)}$ convex.

Since $S = S_1 \cap S_2^{(t)}$ and both S_1 and S_2 are convex, thus S is convex.