EE-588: Homework # 2

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Contents

3.15

a)

$$\begin{aligned}\lim_{\alpha \longrightarrow 0} u_alpha(x) &= \lim_{\alpha \longrightarrow 0} \frac{x^\alpha - 1}{\alpha} \\ &= \lim_{\alpha \longrightarrow 0} \frac{\frac{d}{d\alpha}(x^\alpha - 1)}{\frac{d\alpha}{d\alpha}} \quad (l'Hopital's rule) \\ &= \lim_{\alpha \longrightarrow 0} x^\alpha \log(x) \\ &= \log(x)\end{aligned}$$

b)

$$\begin{aligned}\nabla u_\alpha(x) &= x^{\alpha-1} \\ \implies \nabla u_\alpha(x) &> 0 \quad (x \in \mathbb{R}_+) \\ \nabla^2 u_\alpha(x) &= \alpha(\alpha-1)x^{\alpha-2} \\ \implies \nabla^2 u_\alpha(x) &\leq 0 \quad (0 \leq \alpha \leq 1)\end{aligned}$$

Also,

$$\begin{aligned}u_\alpha(1) &= \frac{1^\alpha - 1}{1} \\ &= 0\end{aligned}$$

3.16

c)

$$\nabla f = \begin{pmatrix} \frac{-1}{x_1^2 x_2} \\ \frac{-1}{x_1 2 x_2^2} \end{pmatrix}$$
$$\nabla^2 f = \begin{pmatrix} \frac{2}{x_1^3 x_2} & \frac{1}{x_1^2 x_2^2} \\ \frac{1}{x_1^2 x_2^2} & \frac{2}{x_1 x_2^3} \end{pmatrix}$$

Now,

$$\text{On } \mathbb{R}_+ +^2 : \frac{2}{x_1^3 x_2} \geq 0; \frac{2}{x_1 x_2^3} \geq 0; \frac{2}{x_1^3 x_2} \times \frac{2}{x_1 x_2^3} \geq (\frac{1}{x_1^2 x_2^2})^2$$

, i.e. $\nabla^2 f(x)$ is positive semidefinite.

$f(x)$ convex and quasiconvex.

d)

$$f(x_1, x_2) = \frac{x_1}{x_2}$$
$$\nabla f = \begin{pmatrix} \frac{1}{x_2} \\ -\frac{x_1}{x_2^2} \end{pmatrix}$$
$$\nabla^2 f = \begin{pmatrix} 0 & \frac{-1}{x_2^2} \\ \frac{-1}{x_2^2} & \frac{2x_1}{x_2^3} \end{pmatrix}$$

$\nabla^2 f$ is neither positive nor negative semidefinite.

$f(x)$ is neither convex nor concave.

e)

$$f(x_1, x_2) = x_1^2 / x_2$$
$$\nabla f = \begin{pmatrix} 2x_1/x_2 \\ \frac{-x_1^2}{x_2^2} \end{pmatrix}$$
$$\nabla^2 f = \begin{pmatrix} \frac{2}{x_2} & \frac{-2x_1}{x_2^2} \\ \frac{-2x_1}{x_2^2} & \frac{2x_1^2}{x_2^3} \end{pmatrix}$$
$$= \frac{2}{x_2} \begin{pmatrix} 1 \\ \frac{-2x_1}{x_2} \end{pmatrix} \begin{pmatrix} 1 & \frac{-2x_1}{x_2} \end{pmatrix} \succcurlyeq 0$$

Now,

$$\text{For } x \in \mathbb{R}_+ + : 2/x_2 > 0; 2x_1^2/x_2^2 \geq 0; \frac{2}{x_2} \times \frac{2x_1^2}{x_2^3} \geq (\frac{-2x_1}{x_2^2})^2$$

$f(x)$ is convex and quasiconvex.

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f)

$$
\begin{aligned}
f(x_1, x_2) &= x_1^{\alpha} x_2^{1-\alpha} \\
\nabla f &= \begin{pmatrix} \alpha x_1^{\alpha-1} x_2^{1-\alpha} \\ (1-\alpha) x_1^{\alpha} x_2^{-\alpha} \end{pmatrix} \\
\nabla^2 f &= \alpha(\alpha-1) x_1^{\alpha} x_2^{-\alpha} \begin{pmatrix} -x_1^{-2} & x_1^{-1} x_2^{-1} \\ x_1^{-1} x_2^{-1} & -x_2^{-2} \end{pmatrix} \\
&= -\alpha(1-\alpha) x_1^{\alpha} x_2^{1-\alpha} \begin{pmatrix} x_1^{-2} & -x_1^{-1} \\ -x_1^{-1} x_2^{-1} & x_2^{-2} \end{pmatrix} \preccurlyeq 0
\end{aligned}
$$

Not convex or quasiconvex.

3.22

a)

$$f(x) = -\log(-\log(\sum_{i=1}^{n} e^{a_i^T x + b_i}))\text{on dom } f = \{x|\sum_{i=1}^{n} e^{a_i^t x + b_i} N1\}$$

Given $\log(\sum_{i=1}^n e^{y_i})$ is convex. Hence, $\log(\sum_{i=1}^n e^{a_i^T x+b_i})$ is also convex as it is an affine transformation of x_i and hence $-\log(\sum_{i=1}^n e^{a_i^T x+b_i})$ is concave. $\log(y)$ is concave for concave y and hence $\log(-\log(\sum_{i=1}^n e^{a_i^T x+b_i}))$ is concave and hence $f(x) = -\log(-\log(\sum_{i=1}^n e^{a_i^T x+b_i}))$ is convex.

b)

$$\begin{aligned}
f(x,u,v) &= -\sqrt{uv - x^T x} \text{ on dom } f = \{(x,u,v)|uv > x^T x, u, v > 0\} \\
&= -\sqrt{u(v - x^T x/u)} \\
&= -\sqrt{uz} \quad (z = v - x^T x/u > 0; u > 0) \\
\nabla f(uz) &= -\frac{1}{2}\begin{pmatrix} \left(\frac{z}{u}\right)^{-\frac{1}{2}} \\ \left(\frac{u}{z}\right)^{-\frac{1}{2}} \end{pmatrix} \\
\nabla^2 f(uz) &= \frac{1}{4\sqrt{zu}}\begin{pmatrix} \frac{z}{u} & 1 \\ 1 & \frac{z}{u} \end{pmatrix} \\
\frac{z}{u} &= \frac{v - x^T x/u}{u} \\
&> 0
\end{aligned}$$

and hence $f(x,u,v)$ is convex.

c)

$$\begin{aligned}
f(x,u,v) &= -\log(uv - x^T x) \text{ on dom } f = \{(x,u,v)|uv > x^T x, u, v > 0\} \\
&= -\log(u) - \log(v - x^T x/u) \\
&= -\log(u) - \log(z) \quad (z = v - x^T x/u > 0; u > 0)
\end{aligned}$$

$-\log(u)$ and $-\log(z)$ are both convex for $u, z > 0$ and hence $f(x,u,v)$ is convex.

d)

$$\begin{aligned}
f(x,t) = -(t^p - ||x||_p^p)^{1/p} \text{ on dom } f = \{(x,t)|t \geq ||x||_p\} \\
&= -(t^{p-1}(t - ||x||_p^p/t^{p-1})^{1/p} \\
&= -(t^{1-1/p})(t - ||x||_p^p/t^{p-1})^{1/p} \\
&= -y^{1/p} z^{1/p} \quad (y = t^{p-1}, z = t - ||x||_p^p/t^{p-1})
\end{aligned}$$

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e)

$$\begin{aligned} f(x,t) &= -\log(t^p - ||x||_p^p) \quad (p > 1 \text{ and on dom } f = \{(x,t)|t > ||x||_p\} \\ &= -\log(t^{p-1}(t - ||x||_p^p/t^{p-1})) \\ &= -(p-1)\log(t) - \log((t - ||x||_p^p/t^{p-1})) \end{aligned}$$

$-(p-1)\log(t)$ is convex. and $\log((t - ||x||_p^p/t^{p-1}))$ is concave as $t - ||x||_p^p/t^{p-1}$ is concave and hence -$\log((t - ||x||_p^p/t^{p-1}))$ is convex resulting in $f(x,t)$ convex.

3.49

a)

$$f(x) = \frac{e^x}{1+e^x}$$
$$\log(f(x)) = \log(e^x) - \log(1+e^x)$$
$$= x - \log(1+e^x)$$

x is concave and $\log(1+e^x)$ is convex as its is log-sum-exp. Hence $x-\log(1+e^x)$ is concave $\implies f(x)$ is log-concave

d)

$\log(f(x) = \log(\text{det}(X)) - \log(\text{tr}(X))$

We first prove the log-concavity of the determinant:

Define $g(t) = \log(X+tV)$ where $X+tV$ is a positive definite matrix. X is given to be positive definite so $\exists$

$$f(t) = \log(\text{det}(X^{\frac{1}{2}}X^{\frac{1}{2}} + tX^{\frac{1}{2}}X^{\frac{-1}{2}}VX^{\frac{-1}{2}}X^{\frac{1}{2}}))$$
$$' = \log(\text{det}(X^{\frac{1}{2}}(I+tX^{\frac{-1}{2}}VX^{\frac{-1}{2}})X^{\frac{1}{2}}))$$
$$= \log(\text{det}(X^{\frac{1}{2}})) + \log(\text{det}(I+tX^{\frac{-1}{2}}VX^{\frac{-1}{2}})) + \log(\text{det}(X^{\frac{1}{2}}))$$
$$= \log(\text{det}(X)) + \log(\text{det}(I+tX^{\frac{-1}{2}}VX^{\frac{-1}{2}}))$$

Since X and $X+tV$ are positive definite, $X^{\frac{1}{2}}$ and $tX^{\frac{-1}{2}}VX^{\frac{-1}{2}}$ are positive definite.

Let $\lambda_1, \lambda_2, \lambda_d$ be eigen values of $X^{\frac{-1}{2}}VX^{\frac{-1}{2}}$ such that $X^{\frac{-1}{2}}VX^{\frac{1}{2}} = Q\Lambda Q^T = \sum_{i=1}^d \lambda z_i z_i^T$

$$\log(\text{det}(I+tX^{\frac{-1}{2}}VX^{\frac{-1}{2}})) = \log(\prod_{i=1}^d (1+t\lambda_i))$$
$$= \sum_{i=1}^d \log(1+t\lambda_i)$$
$$\implies g(t) = \log(\text{det}(X)) + \sum_{i=1}^d \log(1+t\lambda_i)$$
$$g''(t) = -\sum_{i=1}^d \frac{\lambda_i^2}{(1+t\lambda_i)^2}$$
$$\leq 0$$

Hence, $\log(\text{det}(X))$ is concave.

Now consider $\log(\text{tr}(X+tV))$:

$$\log(\text{tr}(X+tV)) = \log(\text{tr}(X(I+tX^{\frac{-1}{2}}VX^{\frac{1}{2}})))$$
$$= \log(\sum_{i=1}^d z_i^T X z_i (1+t\lambda_i))$$

which is concave.

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Combining,

$$\begin{aligned}\log(\mathsf{det}(X+tV)) - \log(\mathsf{tr}(X+tV)) &= \log(\mathsf{det}(X)) + \sum_{i=1}^{d}\log(z_i^T X z_i)(1+t\lambda_i) - \sum_{i=1}^{d}\log((z_i^T x z_i)(1+t\lambda_i)) \\ &= \log(C) + \sum_{i=i}^{n}\log(x_i) - \log(\sum_{i=1}^{n} x_i) \quad \big(x_i = z_i^T X z_i(1+t\lambda_i); C \text{ constant}\big)\end{aligned}$$

$\sum_{i=i}^{n}\log(x_i) - \log(\sum_{i=1}^{n} x_i)$ is concave.

4.11

a) Minimize $||Ax - b||_\infty$:
Equivalent LP:

$$\text{minimize } t$$
$$\text{subject to } Ax - b \geq\succcurlyeq -t\mathbf{1}$$
$$Ax - b \geq\preccurlyeq t\mathbf{1}$$

How equivalent:
For $x = (x_1, \dots x_n)$ (x_i is scalar) and $A = (a_1, a_2, \dots, a_n)$ (a_i are columns) we have:
$a_k^T x_k + b_k \geq -t$ and $a_k^T x_k + b_k \leq t \implies |a_k^T x - b_k| \leq t \implies t \geq max|a_k^T x - b_k| = ||Ax - b||_\infty$. The optimal value solution for this is $p^*(x) = ||Ax - b||_\infty$ as desired.
b) Minimize $||Ax - b||_1$

$$\text{minimize } \mathbf{1}^T t$$
$$\text{subject to } Ax - b \geq\succcurlyeq -t$$
$$Ax - b \geq\preccurlyeq t\mathbf{1}$$

Equivalence:
From the constraint: $|a_k^T x_k - b_k| \leq t_k$ for each k. Hence the optimum is given by $s_k = |a_k^T x_k - b_k| \implies p^*(x) = ||Ax - b||$
c) Minimize $||Ax - b||_1$ subject to $||x||_\infty \leq 1$:
Equivalent LP: $x \in \mathbb{R}^n$ and $t \in \mathbb{R}^m$

$$\text{minimize } \mathbf{1}^T t$$
$$\text{subject to } t - 1 \leq 0$$
$$-1 - t \leq 0$$
$$Ax - b \leq t$$
$$-Ax + b \leq -t$$

Equivalence is implied from the previous two cases (a) and (b).
d) Minimize $||x||_1$ subject to $||Ax - b||_\infty \leq 1$
Equivalent LP:

$$\text{minimize } \mathbf{1}^T t$$
$$\text{subject to } Ax - b \leq \mathbf{1}$$
$$-Ax + b \leq -\mathbf{1}$$
$$-1 \leq x \leq t$$

e) Minimize $|Ax - b||_1 + ||x||_\infty$
Equivalent LP:

$$\begin{aligned} \text{minimize } & \mathbf{1}^T t + y \\ \text{subject to } & -t \leq Ax - b \leq t \\ & -y\mathbf{1} \leq x \leq y\mathbf{1} \end{aligned}$$

Additional Exercise

(a) Minimize $f(x) = c^T F(x)^{-1} c$ where $c \in \mathbb{R}^m$. Making use of Schur's complement, we have an equivalent LP:

$$\min t$$

$$\text{such that} \begin{pmatrix} F(x) & c \\ c^T & 1 \end{pmatrix} \succcurlyeq 0$$

(b) Minimize $f(x) = \max_{i=1,\ldots,K} c_i^T F(x)^{-1} c_i$ where $c_i \in \mathbb{R}^m$ where $i = 1, \ldots, K$
Again, making use of Schur's complement as in (a):

$$\min t$$

$$\text{such that} \begin{pmatrix} F(x) & c_i \\ c_i^T & 1 \end{pmatrix} \succcurlyeq 0 \quad i = 1, \ldots, K$$

(c) Minimize $f(x) = \sup_{||c||_2 \leq 1} c^T F(x)^{-1} c$
Let $F(x)^{-1} = Q\Lambda Q^T$, thus $\sup_{||c||_2 \leq 1} c^T F(x)^{-1} c = c^T Q \Lambda Q^T c$. Thus, $\min f(x) = \lambda_{\max}(F(x)^{-1})$.
$f(x) \leq t \implies \lambda_{\max}(F(x)^{-1}) \leq t \implies F(x)^{-1} \preccurlyeq tI$.Now making use of Schur's complement:

$$\min t$$

$$\begin{pmatrix} F(x) & I \\ I & tI \end{pmatrix} \succcurlyeq 0$$

(d)

$$
\begin{aligned}
f(x) &= E[c^T F(x)^{-1} c] \\
E[C] &= \bar{c} \\
E[(c-\bar{c})(c-\bar{c})] &= S \\
S &= E[cc^T] - \bar{c}\bar{c}^T \\
E[c^T F(X)^{-1} c] &= E[(c-\bar{c}+\bar{c})^T F(X)^{-1}(c-\bar{c}+\bar{c})] \\
&= E[(c-\bar{c})F(X)^{-1}(c-\bar{c})] + E[\bar{c}^T F(x)^{-1} c] \\
&= E[(c-\bar{c})F(X)^{-1}(c-\bar{c})] + \bar{c}^T F(x)^{-1}\bar{c} \quad \because E[c] = \bar{c} \\
&= \text{tr}(F(x)^{-1} S) + \bar{c}^T F(x)^{-1} \bar{c}
\end{aligned}
$$

Let $S = \sum_{i=1}^d s_i s_i^T$ so that the problem becomes equivalent t to minimize $\bar{c}^T F(X)^{-1}\bar{c} + \sum_{i=1}^d s_i^T F(X)^{-1} s_i$
Equivalent LP:

$$
\begin{aligned}
\text{minimize } & t_0 + \sum_{i=1}^d t_i \\
\text{subject to } & \begin{pmatrix} F(x) & \bar{c} \\ \bar{c}^T & t_0 \end{pmatrix} \succcurlyeq 0 \\
& \begin{pmatrix} F(x) & s_i \\ s_i^T & t_0 \end{pmatrix} \succcurlyeq 0 \quad (i = 1, 2, \ldots, d)
\end{aligned}
$$